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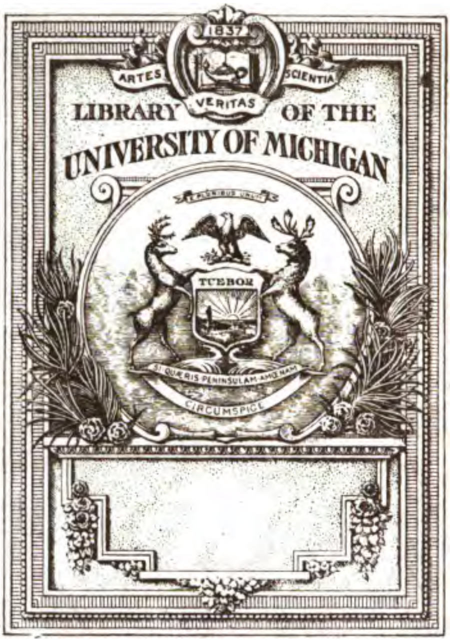
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ON THE
MOTION OF POINTS
CONSTRAINED AND RESISTED,
AND ON THE
MOTION OF A RIGID BODY.

THE SECOND PART OF A NEW EDITION

OF

A TREATISE ON DYNAMICS.

By WILLIAM WHEWELL, M.A.
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PREFACE.

In preparing for the press a new edition of the concluding portion of my Treatise on Dynamics, I have endeavoured to correct the errors of the former edition, and, in many instances, to simplify and improve the work. I have, as in the former part of the Treatise, represented the integration of an expression by $\int_x$ prefixed to the differential *coefficient*, (taken with respect to x,) and have in all cases used differential coefficients only, to the exclusion of differential quantities. I need not explain how much this restriction removes the chance of the error or confusion into which the young analyst may be led when differentials are taken on various hypotheses of other differentials being constant or variable. For the differential coefficients of y, I have used the common notation, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, &c. each of which apparent fractions must be considered as a single symbol. The square of $\frac{dy}{dx}$ ought therefore to be represented by $\left(\frac{dy}{dx}\right)^2$; but in some instances I have, for the sake of convenience, used the notation $\frac{dy^2}{dx^2}$ instead of this. I may observe, that in using the differential coefficient, we must take for granted a complete familiarity with all such steps of reasoning as the following:

$$\frac{dv}{dt} = P; \text{ therefore } \frac{dt}{dv} = \frac{1}{P}, \text{ also } \frac{dv}{dx} = P\frac{dt}{dx};$$

$$\int_s Q = \int_v Q\frac{ds}{dv}; \text{ and the like.}$$

I have also modified, in some measure, the statement of the principles introduced in this volume. The only place in which any new principles are needed, in addition to those which are contained in the former part of the work, is in the problems respecting the motion of a rigid body. I have now presented the fundamental principle on which the solution of all such problems must depend, as an extension of the third law of motion: this is, I conceive, the true nature of this principle, in whatever way it be stated;—whether as an assertion of the equivalence of the impressed and effective forces, or of the equality of action and reaction, or of the equilibrium of forces lost, or in whatever other manner.

For the purpose of making the present volume more independent, and also with the hope of pointing out the logical connexion of the fundamental principles of mechanics, I will place at the end of this preface a brief statement of these principles.

It will be found that in this statement they are made to depend upon certain general principles respecting cause and effect, action and reaction. To some persons it may appear that any reference to such general notions as these must necessarily be vague and barren; but to this we may reply, that the generality of these notions is the condition and foundation of the generality of our scientific knowledge; that if we did not recognize the universal truth of the axioms concerning cause and effect, action and reaction, it would be impossible for us to have any knowledge concerning the laws by which force and matter are universally governed. Particular facts could not produce general knowledge, if it were not that the constitution of our minds leads us to apprehend particulars as cases of generals; and this constitution exhibits itself in the notions of cause and effect, action and reaction.

Other persons may be disposed to say, that such axioms as we are now referring to are merely definitions;—that they assert nothing more than the meanings of the terms employed. Or this we may remark, that definitions, as well as axioms, undoubtedly may be the basis of demonstration, as we see in geometry: and if it be possible to give such definitions of the terms we employ as shall lead to valuable truths, there can be no better starting point for our reasonings than such definitions are. But when men assert that any proposition is a definition merely, they commonly mean to imply that it is an arbitrary definition, and that any other might be assumed with as good right and reason. Now this is not the case with the assertions here referred to. If they are merely an explanation of the meaning of certain terms, it still remains for us to ask, how came there to be such a meaning in men's minds? Why were such aspects and relations of facts selected, rather than any others, to have names imposed on them? What peculiarity in these notions induced men to set about defining them? And to this it is clear that we must answer, that the notions of cause, and of mutual action, are important, because we can obtain important truths by introducing them into our reasonings;—that their efficacy in this respect appears to depend on a preeminence of simplicity, or generality, or symmetry, in the relations they contain; and that thus, as we have already said, they carry our minds to that mode of viewing the world of matter and motion, by means of which general science concerning it becomes possible.

Other persons again may maintain that these axioms concerning cause, and the properties of matter, are mere results of experience; and that therefore such principles cannot elevate mechanics above the rank of an empirical science: they may add that we cannot suppose a person to be possessed of the notions of cause, and of matter, except as a consequence of his observation of the external world. This assertion however employs the terms experience and observation in a sense different from that which is usually implied in such enquiries:

for when we speak of knowledge derived from experience, it is generally understood that we admit the possibility of knowledge of another kind: and the distinction of such two kinds of knowledge is important; for the knowledge which we obtain from special and contingent experience, (the facts of natural history for example) is altogether different from the necessary and universal truths which we cannot avoid assenting to, (as the propositions of Geometry.) If any one asserts that the axioms of Geometry are results of experience, he must also assert that we learn by experience that eight and nine make seventeen: such *experience*, if any one will call it so, is inevitable and necessary in the nature of man; we cannot speak in general terms, nor even think, without assuming its results; and to say that we could not know any particular truth without *such* experience, is to say that we could not know without the faculty of perceiving. On this fundamental and inevitable experience, that is, on our powers of perception as well as our powers of thought, depend those general axioms concerning cause and matter which are the foundation of the doctrines of mechanics. But there is another kind of experience, to which the name is commonly applied, which is concerned in building up these doctrines, in a manner altogether different from this primitive and universal conviction; namely, the subsequent and distinctive experience which teaches us the character and measure of particular facts: and this experience, to which alone the name can properly be applied, is requisite, in order to give a definite import and meaning to the terms which our general axioms involve. This experience is distinctive; it implies not only the faculty of perceiving, but special objects perceived; not merely the perception of something, but the perception that things are and occur in a certain manner to the exclusion of any other manner. It differs from our primitive and necessary conviction, being not only subsequent, but contingent; it not only comes after the first exercise of perception, but it may fail to come at all; the ancients, and many of the moderns, have been destitute of that experience which proves our laws of motion to be true, and have accordingly believed them to be false. And the view of the nature of mechanical principles which is here maintained, is this;—that the generality of these principles and of the terms in which they are expressed, is obtained from the fundamental and universal suggestion of a sort of experience, (which we may call *connate experience*, since our convictions are produced along with our ideas); or rather it depends upon the general notions of cause and mutual action, under which alone we can speculate concerning the objects of experience: but that for the special import and measures of these terms, we must recur to a special, distinctive, and contingent experience, such as the name generally implies.

We may add, that when we clothe our special experience in the terms which refer to the necessary conditions of our knowledge, we may make it appear to some, that the substance as well as the conditions, the matter as well as the form of our knowledge, is *necessarily* such as it is: but that the error of such a judgment is manifestly shewn by the differences of opinion which have prevailed, and prevail even yet, concerning mechanical truths, among persons who acknowledge the truth of the principles of mechanics so far as they are included in those axioms; those axioms being in fact acknowledged by all reasoners, as the universal formulæ of speculative truth.

I proceed to state briefly the First Principles of Mechanics and the foundations on which they rest.

ELEMENTARY PRINCIPLES

OF

MECHANICS.

GENERAL PRINCIPLES.

1. MECHANICS is the science which treats of the motion of bodies, with reference to its causes, its laws, and the consequences of the laws.

It is the object of science to reduce phenomena to their causes: the fundamental principle of science is

PRINCIPLE I. *No change can take place without a cause.*

2. The cause of the motion (or prevention of motion) of bodies, is *force*.

Force is considered as a quality by which one body acts upon another; its intensity is supposed to depend upon the position of the body acted on, with regard to the acting body, and to be measurable in numbers.

The intensity of a cause must be measured by its effects. The following is the principle on which this measurement proceeds.

PRINCIPLE II. *Effects are proportional to their causes.*

3. When the first body acts upon the second, the second *reacts* upon the first: a body cannot be pressed without exerting a resisting pressure; when a body receives a blow, it by the same act gives one. Such a reaction is inseparable from our conception of body or matter.

The Action and Reaction are forces of the same kind, and each determines the other according to the same law. This gives rise to the following principle.

PRINCIPLE III. *The action of one body upon another is accompanied by an equal and opposite reaction.*

4. Force exerted in the action and reaction of bodies is called *Pressure.*

The idea of pressure is contained in our elementary sensations and observations, and must be pre-supposed in all mechanical reasoning.

The *Weight* of bodies is their constant and universal pressure downwards.

5. Pressure may be employed in producing *rest* or *motion:* the branch of mechanics which treats of the first case is called *Statics*; that which treats of the case in which pressure produces motion, is called *Dynamics.*

PRINCIPLES OF STATICS.

6. In Statics, forces are said to balance each other, or to be in equilibrium. A simple case of equilibrium, of which the consideration is a sufficient foundation for the whole of Statics, is that of the *Lever*

A Lever is a rod which, being moveable about a fixed point called the *Fulcrum,* is acted upon by forces tending to turn it round this point.

7. In commencing our reasonings, the lever is supposed to be without weight, and therefore the effect of forces which act upon it, depends entirely upon the magnitude of the forces, and upon their position. Hence the following axiom is true;

AXIOM I. *If on a horizontal straight lever equal weights be fixed at equal distances from the fulcrum on opposite sides, they will balance each other.*

8. The following is a fundamental principle of statics, the conviction of the truth of which is obtained in obtaining our idea of weight.

If a body consisting of heavy parts be supported by a single force, the requisite force is not altered by altering the arrangement of the parts.

For instance, the force requisite to sustain a basket of stones continues the same, while the stones are shaken into different positions.

The simplest case of this Principle is sufficient to form a foundation for the science, and is stated in the following axiom.

Axiom II. *If two equal weights balance each other upon a straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever.*

We learn also, in the same manner, that the weight of a body, and the direction in which it tends to fall, are not sensibly altered by change of place.

9. From the second axiom we can deduce this corollary.

If a weight be supported upon a lever which rests on two fulcrums at equal distances from the weight, the whole pressure upon the fulcrums is equal to the weight.

From these axioms the properties of the lever may be deduced, and thence all the doctrine of statics.

Principles of Dynamics.

10. The fundamental principles of Dynamics are three, resulting from the application of the three General Principles above stated to the case of Forces.

First Law of Motion. *A body in motion, not acted upon by any force, will move in a straight line with a uniform velocity.*

By the first General Principle, it appears that the velocity or direction of the body's motion cannot change without a cause.

Any cause of change of the direction of a body's motion depending upon its relations to space is called force; and therefore when a body is not acted upon by any force, no such change can occur.

It appears by observation, that the time during which a body has been in motion is not a cause of any change of velocity.

Therefore when a body is not acted on by any force, neither the velocity nor direction of its motion will change.

11. SECOND LAW OF MOTION. *When any force acts upon a body in motion, the change of motion which is produced is in the direction and proportional to the magnitude of the force which acts.*

This law may be divided into two cases.

Case 1. When the force acts in the direction in which the body is always moving.

The force increases the velocity, and therefore by the second General Principle, the magnitude of the force is measured by the increase of velocity.

A force is uniform when its measure, (so far as it depends on the position of the point where the force acts) is always the same; and conversely, if the increase of the velocity be the same in all parts of space, the force is uniform: thus gravity is a uniform force, as to sense.

A uniform force, acting in the direction in which a body is moving, is found to produce the same increase of velocity *in a given time*, whether the body be moving quick or slow, or if it begin to move from rest. Therefore the effect of a uniform force in this case is independent of the rest or motion of the body.

Hence the magnitude of a uniform force is measured by the *velocity which it generates or adds in a unit of time*, as one second; in pursuance of the second General Principle.

The magnitude of a *variable* force is measured by the *limit* of the velocity generated, estimated in proportion to the time.

Hence the Second Law of Motion is true in this case for variable forces also.

The magnitude or quantity of a uniform force, and of a variable force, measured in the manner just stated, is the *accelerating quantity* of the force, and is usually called simply the *Accelerating Force*.

12. *Case* 2. When the force acts transversely or obliquely to the direction in which the body is moving.

When two relative *motions*, represented in direction and velocity by two contiguous sides of a parallelogram, are *compounded*, the result is a motion represented by the diagonal. This results from the definition of *composition of motion*.

When a uniform force acts upon a body already moving, in a direction transverse or oblique to the direction of the motion, it produces the effect of compounding with the motion which the body already possesses, a velocity equal and parallel to that which the same force would produce in a body at rest. This appears by observation; for instance, in bodies falling or thrown in a ship which is under way.

Hence the Second Law of Motion is true for a uniform force.

Also every variable force will coincide with a uniform force in the limit of its effect, if it coincide with the uniform force in its measure, since the measure is obtained by the limit of the effect. Therefore the Second Law of Motion is true for variable forces also.

13. Third Law of Motion. *When pressure produces motion, the effective moving force is as the pressure.*

This is applicable in two cases.

Case 1. When force acts upon a system *directly*, so that the mutual action of two bodies is in the line which joins them.

When a force acts on any system, so long as the pressure is the same, the magnitude of the *Action*, as shewn by the motion produced, is the same also: for force considered as pressure is identical with force considered as the cause of motion.

Two equal pressures, acting for the same time, will put two equal bodies in motion with equal velocities; and these pressures, acting together, make a pressure twice as great as either of them; therefore the action is twice as great. Therefore, the velocity produced in a given time by any force, being given, the amount of *action* is proportional to the quantity of matter moved.

Inertia is the quantity of matter of a body, thus measured by the pressure requisite to communicate to it a given velocity in a given time.

14. The *Momentum* of a body is the product of the numbers which represent its quantity of matter and its velocity.

It appears by observation that, in the direct mutual action of bodies, the momentum gained by the body acted on, and that lost by the acting body *in the same time*, are equal.

For instance, if *A* impinge upon *B* directly, or if *A* communicate motion to *B* by the continuous action of its weight, *B*'s momentum is greater than it would have been in the same time, if *B* had not been connected with *A*, and *A*'s momentum is less than it would have been in the same time, by exactly the same quantity.

Hence, *momentum gained and lost in a unit of time* may be taken as the measure of *action* and *reaction*, in pursuance of the third General Principle.

The *motive quantity* of the force which bodies exert on one another, as measured by the momentum generated in a given time, is called *Moving Force*; and it appears, by what has been said, that this moving force may be taken as the measure of that *Action* which is identical with pressure.

15. *Case* 2. When force acts upon a system by means of any mechanical connexion whatever.

In a system which is in motion, the moving forces, as inferred from the momenta gained and lost by each part of the system, are called the *effective moving forces*, and the pressures which produce motion are called the *impressed forces*. In this case,

The effective forces are equivalent to the impressed forces, according to the statical properties of the system.

This is proved by reasoning similar to that employed in Case 1.

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A

TREATISE

ON

DYNAMICS.

BOOK I.

THE MOTION OF POINTS IN A NON-RESISTING SPACE.

PART II.

THE CONSTRAINED MOTION OF POINTS.

140. It has already been stated, in the commencement of this work, that the motion of points may be free, constrained, or resisted: also that the motion is constrained, when the points of the moving system are subjected to the necessity of being always in a given line, or in a given surface, or at a given distance from one another, or to some similar condition.

In this case, as was then observed, besides the external forces which act upon each point, and which depend upon its position, it is acted upon by other forces also, arising from the resistance of the curves and curve surfaces on which it moves, and from the connection of the points; and by means of these forces the conditions of its constrained motion are satisfied.

CHAP. I.

THE CONSTRAINED MOTION OF A POINT ON A GIVEN LINE OR SURFACE.

141. If we suppose a body, acted on by any forces, to slide along a surface, as the inside of a bowl which is perfectly smooth, the body is perpetually deflected, by the impenetrability of the surface, from the path in which it would move in consequence of the action of the forces alone. Since the surface is supposed perfectly smooth, the force which it exerts upon the body must be, at each point, perpendicular to the surface; for there is no reason why the direction of the action of the surface should be inclined on one side rather than the other, to this perpendicular; if any lateral force do exist, it is attributed to the defect from absolute smoothness, and called *friction*.

This perpendicular force exerted by the surface, varies perpetually during the motion of the body: it is always such as exactly to keep the body in the given surface, that is, to resist the tendency to move *through* the surface. Now, if we were to suppose the body to move freely, and a force of the same magnitude and direction as this reaction, but not arising from the contact of the body with a surface, always to act upon it, its motion would be exactly the same: but in this case, we may calculate the motion by the formulæ of the preceding Chapters, introducing among the forces this new one, and making it such as always to keep the body in the surface: we have therefore a method of treating the problem of the motion of a body upon a surface.

The reaction of the surface does not increase or diminish the velocity of the body. This will be seen when we come to deduce the formulæ for the motion. But it appears to be nearly evident from this consideration; that the force of reaction is entirely employed in deflecting the body. If it were not perpendicular to the direction of the motion, we might resolve it into two, one perpendicular to this direction, which deflects the body from its path; and one in the direction of the motion, which alone would affect the velocity.

142. We may, as the simplest case, consider the body to move on a curve *line*, lying in one plane. This will happen when the original motion and the forces are all in one plane, and that plane also every where perpendicular to the given surface on which the body moves. For then the reaction will be in the same plane, and there will be nothing to deflect the body from this plane. It will therefore be sufficient in this case to consider the motion as taking place on a plane curve. The next simplest case will be when the surface is one of revolution, and all the forces act in planes passing through the axis. We shall afterwards consider any surface whatever, with any forces.

Instead of supposing the body to be retained in a certain geometrical surface by the reaction of an impenetrable material face, we may suppose the body to be retained in the same surface by other means. For instance, if a body be fastened by an inextensible string to a given point, it must move in the surface of a sphere: and the conditions of its motion will be the same as if it moved on a smooth spherical surface: the tension of the string supplying the place of the reaction of the surface. And by supposing the string during its motion to wrap round other curves and surfaces, the surface to which the body is thus confined may become any whatever.

We may also, instead of supposing the body to move *on* a curve, conceive it to move *in* a curvilinear *tube*, indefinitely narrow, the body being considered as a point. The difference between this case and the former one, will be, that in this, the reaction can operate in any direction, whereas in the former case it could only act on one side of the body, namely, in the side in which the material face is. Hence in the tube, if the reaction were to become first nothing and then negative, (so as to change sides with respect to the curve) the body would still be retained; while, in the other case, it would, on this supposition, fly off from the curve and describe a path in free space.

In this manner also we may suppose a body to be compelled to move in *a curve of double curvature.*

Sect. I. THE MOTION OF A POINT ON A PLANE CURVE.

143. PROP. *When a body moves on a curve, acted on by given forces, to determine its velocity.*

Let a body *P* move on the curve *PA*, fig. 1, the curve being referred to the co-ordinates *AM*, *MP*, which are represented by x, y: and let the forces which act upon the body be resolved into X, Y, parallel, respectively, to these co-ordinates and tending to increase them. Besides these forces, we have to consider the reaction of the curve, which is in the direction *PK*, perpendicular to the curve, and which we may represent by *PK*, and may resolve into *PL*, *PH*. If we call the reaction R, we shall have, for the resolved parts in *PL*, and in *PH* parallel to *AM*,

$$R\cdot\frac{PL}{PK} \text{ and } R\cdot\frac{PH}{PK}, \text{ respectively;}$$

or, (since the triangles *PLK* and *TMP* are manifestly similar, *PT* being a tangent),

$$R\cdot\frac{MT}{PT}, \text{ and } R\cdot\frac{MP}{PT}; \text{ that is, } R\frac{dx}{ds}, \text{ and } R\frac{dy}{ds}:$$

supposing $\frac{ds}{dx} = \sqrt{\left(1+\frac{dy^2}{dx^2}\right)}$, and s being therefore the curve *AP*.

Hence, collecting the whole forces in the directions *PH* and *MP*, we have, by equations (a) of Article 19,

$$\frac{d^2x}{dt^2} = X + R\frac{dy}{ds},$$

$$\frac{d^2y}{dt^2} = Y - R\frac{dx}{ds}.$$

Now, to eliminate R, multiply by $2\frac{dx}{dt}$ and $2\frac{dy}{dt}$, respectively, and add; and we have

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}+2\frac{dy}{dt}\frac{d^2y}{dt^2}=2X\frac{dx}{dt}+2Y\frac{dy}{dt};$$

integrating with respect to t

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}=2\int_t\left(X\frac{dx}{dt}+Y\frac{dy}{dt}\right)=2\left(\int_x X+\int_y Y\right).$$

This expression is the same as it would be if the body moved freely. Hence, it appears that when a body, acted on by given forces, moves from one given point to another, as from B to P, the velocity is the same, whatever be the path it takes, and whether it moves freely or be constrained to move along a given curve.

If we suppose the body to be acted on by a force in parallel lines, we may suppose the axis of x to be parallel to these lines; and we have then $Y=0$.

If the force be also supposed to be constant, as for instance, gravity, and x to be measured upwards, we have $X=-g$; and $2\int_x X=C-2gx$. Or, if we put $C=2gh$, h being a quantity depending on circumstances, we have

$$(\text{velocity})^2=\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}=2g(h-x).$$

Here, when $x=h$, the velocity is $=0$; therefore h is the height from which the body begins to fall. Also since the velocity depends on x alone, it appears that it is the same, whether the body fall down the perpendicular DM, or down any curve BP of the same vertical height.

If we suppose the force to tend to a centre, and to vary as some function of the distance from it; the centre of force may be made the origin of co-ordinates, as A, fig. 2. Let $AP=r$, and the force in $PA=P$, a function of r. Hence, we have, since the resolved forces tend to diminish x and y,

$$X=-\frac{Px}{r},\quad Y=-\frac{Py}{r},$$

$$\int_x X+\int_y Y=\int_r\left(X\frac{dx}{dr}+Y\frac{dy}{dr}\right)=\int_r -P\left(\frac{x}{r}\frac{dx}{dr}+\frac{y}{r}\frac{dy}{dr}\right).$$

And since $x^2 + y^2 = r^2$, $\frac{x}{r}\frac{dx}{dr} + \frac{y}{r}\frac{dy}{dr} = 1$;

$$\text{whence } \textstyle\int_x X + \int_y Y = -\int_r P.$$

If s be the length of any arc, the velocity $= \frac{ds}{dt}$:

$$\text{also } \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2},$$

$$\text{Hence (velocity)}^2 = C - 2\textstyle\int_r P.$$

Cor. 1. If the angle $PAM = v$,

$$\frac{ds^2}{dt^2} = \frac{dr^2}{dt^2} + r^2\frac{dv^2}{dt^2};$$

$$\text{whence } \frac{dr^2}{dt^2} + r^2\frac{dv^2}{dt^2} = C - 2\textstyle\int_r P,$$

which is applicable when the force acts to a centre, or P is a function of r.

Cor. 2. Since the velocity depends on r alone, it is the same whether the body fall down the curve BP, or down the line BQ (making $AQ = AP$) acted on by the same force. And if the velocity in the curve and in the straight line AB, be equal at any corresponding equal distances from the centre, they will be equal at any other equal distances*.

144. Having found the velocity in terms of the co-ordinates, or of the radius vector, and knowing mereover the nature of the curve, we can find the time of the motion, as will be seen in the following examples.

$$\text{Since } \frac{ds}{dt} = \text{velocity}, \quad \frac{dt}{ds} = \frac{1}{\text{velocity}};$$

hence, when the force acts parallel to x, we have by last Art.

$$\frac{dt}{ds} = \frac{1}{\sqrt{\{2g(h-x)\}}}.$$

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* *Principia*, Book I, Prop. 40.

And when the force acts to a centre, $\frac{dt}{ds} = \frac{1}{\sqrt{(C - 2\int_r P)}}$.

Prob. I. *Let* AP, *fig.* 1, *be a cycloid with its axis vertical: to determine the motion upon it when the body is acted on by gravity.*

We have here $\frac{dy}{dx} = \sqrt{\frac{2a - x}{x}}$; (*Lacroix*, Art. 102.), where a is the radius of the generating circle;

$$\therefore \ \frac{ds}{dx} = \sqrt{\frac{2a}{x}};$$

$$\frac{dt}{ds} = \frac{dt}{ds}\,\frac{ds}{dx} = -\sqrt{\frac{a}{g}}\cdot\frac{1}{\sqrt{(hx - x^2)}};$$

$$\therefore \ t = C - \sqrt{\frac{a}{g}}\cdot \text{arc}\left(\text{ver. sin.} = \frac{2x}{h}\right)$$

$$= \sqrt{\frac{a}{g}}\cdot\left\{\pi - \text{arc}\left(\text{ver. sin.} = \frac{2x}{h}\right)\right\};$$

since when $x = h$, $t = 0$, and arc $= \pi$.

For the whole time of descent to A, $t = \pi\sqrt{\frac{a}{g}}$.

Cor. 1. Hence, the time of descent is the same whatever be the arc BA.

Cor. 2. If the cycloid be completed, as in fig. 3, the body, after descending from B to A, will ascend, by the velocity acquired, to b which is in the same horizontal line with B: for the height up which the body must ascend to lose the velocity, will be the same as that down which it descended to gain it. And the time of moving up Ab, will equal the time of moving down BA.

When the body comes to b, it will have lost all its velocity: it will then descend to A, and rise again to B, and so go on oscillating for ever, on the supposition that the surface is perfectly smooth.

Cor. 3. By *Lacroix*, Elem. Treat. Art. 103, the evolute to the cycloid $EPAe$ consists of two semi-cycloids, CE, Ce.

Hence, if a body be suspended by a string of proper length, which wraps round the curves CE, Ce, and oscillates, the conditions of the motion of the point P will be the same as those of the motion of a body upon a cycloidal curve, just investigated.

If l be the length of the pendulum, which is necessarily equal to the curve EOC;

$$l = 4a, \text{ and therefore the time in } BA = \frac{\pi}{2}\sqrt{\frac{l}{g}};$$

$$\therefore \text{ time of an oscillation from } B \text{ to } b = \pi\sqrt{\frac{l}{g}}.$$

Cor. 4. The time which a body would employ in falling down the vertical length l is $\sqrt{\frac{l}{g}}$;

$$\therefore \text{ time of oscillation : time down pendulum :: } \pi : 1.$$

The semi-cubical parabola, with its axis vertical, is another case in which we can integrate the expression for the time.

Prob. II. *Let AQ, fig. 3, be a circle whose radius is c: the body being acted on by gravity.*

$$y = \sqrt{(2cx - x^2)}; \quad \frac{ds}{dx} = \frac{c}{\sqrt{(2cx - x^2)}};$$

$$\frac{dt}{dx} = \frac{dt}{ds}\frac{ds}{dx} = -\frac{1}{\sqrt{\{2g(h-x)\}}}\frac{ds}{dx}$$

$$= -\frac{c}{\sqrt{2g}} \cdot \frac{1}{\sqrt{\{(h-x)(2cx-x^2)\}}};$$

$$= -\frac{c}{\sqrt{2g}} \cdot \frac{1}{\sqrt{\{(hx-x^2)(2c-x)\}}};$$

We might integrate by expanding $(2c-x)^{-\frac{1}{2}}$. But we may do it more readily in the following manner:

$$\text{Let } \theta = \text{arc}\left(\sin. = \sqrt{\frac{x}{h}}\right); \quad \therefore \frac{d\theta}{dx} = \frac{1}{2\sqrt{(hx-x^2)}}.$$

Also since $\sqrt{\frac{x}{h}} = \sin.\theta$, $x = h\sin.^2\theta$, $2c - x = 2c - h\sin.^2\theta$ $= 2c(1 - \delta^2\sin.^2\theta)$: putting $\delta^2 = \frac{h}{2c}$. Hence,

$$\frac{dt}{d\theta} = \frac{dt}{dx}\frac{dx}{d\theta} = -\frac{c}{\sqrt{(4gc)}} \cdot \frac{2}{\sqrt{(1 - \delta^2\sin.^2\theta)}}$$

$$= -\sqrt{\frac{c}{g}} \cdot \frac{1}{\sqrt{(1 - \delta\sin.^2\theta)}}.$$

The integral of this is an elliptic transcendent, and may be found by the methods of approximation, or the tables, given by Legendre for such functions. The integral must be taken from $x = h$, that is, from $\theta = \frac{\pi}{2}$: and for the whole arc *AP*, it must be taken to $x = 0$, and therefore $\theta = 0$.

If the oscillations be small, h is small; and therefore δ is small. In this case, we may approximate by expanding. We have thus

$$\int_\theta \frac{1}{\sqrt{(1 - \delta^2\sin.^2\theta)}} = \int_\theta \left\{1 + \frac{\delta^2}{2}\sin.^2\theta + \frac{1.3}{2.4}\delta^4\sin.^4\theta + \&c.\right\}.$$

Now, to find $\int\sin.^m\theta d\theta$, where m is an even number, we have (*Lacroix*, Elem. Treat. 205.)

$$\int_\theta \sin.^m\theta = -\frac{1}{m}\cos.\theta\sin.^{m-1}\theta + \frac{m-1}{m}\int_\theta \sin.^{m-2}\theta;$$

and taking the integral from $\theta = 0$ to $\theta = \frac{\pi}{2}$, the first term vanishes; and

$$\int_\theta \sin.^m\theta = \frac{m-1}{m}\int_\theta \sin.^{m-2}\theta \left\{\begin{matrix}\theta = 0 \\ \theta = \frac{\pi}{2}\end{matrix}\right\}.$$

Similarly, $\int_\theta \sin.^{m-2}\theta = \frac{m-3}{m-2}\int_\theta \sin.^{m-4}\theta \left\{\begin{matrix}\theta = 0 \\ \theta = \frac{\pi}{2}\end{matrix}\right\}$;

and so on to $\int_\theta \sin.^2\theta = \frac{1}{2}\int_\theta = \frac{1}{2}.\frac{\pi}{2}$.

Hence, $\int_\theta \sin.^m\theta = \frac{(m-1)(m-3)\ldots\ldots 1}{m.(m-2)\ldots\ldots\ldots 2}\cdot\frac{\pi}{2}$,

and $\int_\theta \frac{1}{\sqrt{(1-\delta^2\sin.^2\theta)}}$, from $\theta = 0$ to $\theta = \frac{\pi}{2}$, is the same as $\int_\theta -\frac{1}{\sqrt{(1-\delta^2\sin.^2\theta)}}$, from $\theta = \frac{\pi}{2}$ to $\theta = 0$. Hence, we have

$$t = \sqrt{\frac{c}{g}}\left\{1 + \left(\frac{\delta}{2}\right)^2 + \left(\frac{1.3.\delta^2}{2.4}\right)^2 + \left(\frac{1.3.5.\delta^3}{2.4.6}\right)^2 + \&c.\right\}.\frac{\pi}{2}.$$

If we neglect all the terms after the first, we have

$$t = \pi\sqrt{\frac{c}{4g}}:$$

which is the time of descent down a cycloidal arc of which the radius is $\frac{c}{4}$. Hence, if in fig. 3, $AF = \frac{1}{2}AC$ be the axis of a cycloid AE; and if PA be a very small arc of this cycloid, and QA a small arc of the circle described with the center C, the times in PA, QA are equal, as appears by comparing this with the result in the last Problem. In fact, C is the centre of curvature of both, and they may, for small arcs, be supposed to coincide.

If we take two terms, we have, since $\delta^2 = \frac{h}{2c}$,

$$t = \frac{\pi}{2}\sqrt{\frac{c}{g}}\left\{1 + \frac{\delta^2}{4}\right\} = \frac{\pi}{2}\sqrt{\frac{c}{g}}.\left\{1 + \frac{h}{8c}\right\}.$$

But if AB, the chord of the whole descent $= k$, we have

$$h = \frac{k^2}{2c};\quad \therefore\ t = \frac{\pi}{2}\sqrt{\frac{c}{g}}\left\{1 + \frac{k^2}{16c^2}\right\}.$$

Hence, if a pendulum oscillate on each side of the vertical through an arc, the chord of which is $\frac{1}{10}$ of the length of the string, $\frac{k}{c} = \frac{1}{10}$, and the oscillation is longer by $\frac{1}{1600}$ of the whole than it is in an indefinitely small arc.

Prob. III. *Let the curve be the hypocycloid, and let the force tend to the centre of the globe, and be as the distance from the centre.*

A hypocycloid is a curve, *APD*, fig. 4, generated by a point *P*, in the circumference of a circle *GPe*, which rolls along the inside of the circumference *EFD* of another circle. The circle *EFD* is called a circle of *the globe*, and the rolling circle *FPG* is called the *wheel* or generating circle.

Suppose, that when the describing point *P* was at *A*, the diameter *Pe* of the generating circle was in the position *EA*, coinciding with *CE*. And suppose that when *A* comes to *P*, *E* comes to *e*, so that $FE = Fe$. Now when the point is at *P*, the circle is turning on the point *F*; hence, the motion of *P* at that point will be perpendicular to *FP*; and hence, a tangent at *P* will be at right angles to *FP*, and will therefore pass through the point *G*. Let *CY* be a perpendicular on *PG*. And let $CF = a$, $GF = 2b$; $CP = r$, $CY = p$.

Now by similar tringles, $\frac{PY^2}{CF^2} = \frac{GY^2}{CG^2}$;

$$\text{or } \frac{r^2 - p^2}{a^2} = \frac{(a - 2b)^2 - p^2}{(a - 2b)^2} = \frac{e^2 - p^2}{e^2};$$

putting $e = a - 2b = CA$.

$$\text{Hence, } p^2 = \frac{e^2 (a^2 - r^2)}{a^2 - e^2}.$$

(See Mr Peacock's Collection of Examples, p. 195.)

$$\text{Also, } r^2 - p^2 = \frac{a^2 (r^2 - e^2)}{a^2 - e^2}.$$

$$\text{Now, we have in spirals, } \frac{ds}{dr} = \frac{r}{\sqrt{(r^2 - p^2)}}$$

$$= \frac{r\sqrt{(a^2 - e^2)}}{a\sqrt{(r^2 - e^2)}} \text{ in this case.}$$

And, $P = mr$; $\int_r P = \frac{mr^2}{2}$; $C - 2\int_r P = m(h^2 - r^2)$,

making $C = mh^2$; where h is the height CB, when the body begins to move.

Hence,

$$\frac{dt}{dr} = \frac{dt}{ds}\frac{ds}{dr} = -\frac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} \cdot \frac{r}{\sqrt{\{(r^2 - e^2)(h^2 - r^2)\}}},$$

to integrate this, let $\sqrt{(r^2 - e^2)} = u$; $\therefore \frac{r}{\sqrt{(r^2 - e^2)}} = \frac{du}{dr}$;

and $h^2 - r^2 = h^2 - e^2 - u^2$,

$$\frac{dt}{du} = \frac{dt}{dr}\frac{dr}{du} = -\frac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} \cdot \frac{1}{\sqrt{(h^2 - e^2 - u^2)}},$$

$$t = \frac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} \text{ arc } \left(\cos. = \frac{u}{\sqrt{(h^2 - e^2)}} = \frac{\sqrt{(r^2 - e^2)}}{\sqrt{(h^2 - e^2)}}\right),$$

which is $= 0$, when $r = h$, as it should be.

Hence, the time of falling to A, found by making $r = e$, is

$$t = \frac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} \frac{\pi}{2}.$$

Cor. 1. Since h does not appear in this result, the time of descending to A is the same, whatever be the arc; that is, the descents are *isochronous**.

Since $e = a - 2b$, $a^2 - e^2 = 4ab - 4b^2$;

$$\therefore t = \pi\sqrt{\left(\frac{b}{am} - \frac{b^2}{a^2m}\right)}.$$

Cor. 2. If am = force at E, be put $= g$, and if we suppose a large compared with b, so that $\frac{b}{a}$ may be rejected, we have

$$t = \pi\sqrt{\frac{b}{g}},$$

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* *Principia*, Book I, Prop. 51.

which coincides with the time in a common cycloid (see Prob. I.), as it manifestly should; for if a be very large, ED will be nearly a straight line; and the force in DA nearly constant and parallel to EC.

Cor. 3. Hence, also the *oscillations* in such a curve are isochronous, and the time is found as above. The time of an oscillation $= \dfrac{\pi\sqrt{(a^2-e^2)}}{a\sqrt{m}}$.

If we suppose the Earth spherical and homogeneous, the force, in proceeding from the surface to the centre, varies as the distance from the centre. Hence, if we were to suppose a body to move on a hypocycloid, generated by the rolling of a circle on the interior of the circumference of a circle within the Earth, and concentric with it, the descents and oscillations in such a curve would be isochronous.

Cor. 4. Instead of supposing the body to move on a curve, we may suppose it to be suspended by a string, which, during its oscillations, wraps round the curves SD, Sd, the evolutes to the portions AD, Ad of the hypocycloid. The motions will then be the same as before, and the oscillations of such a pendulum will therefore be isochronous.

Cor. 5. *The evolutes SD, Sd are also hypocycloids**.

Let PO be the radius of curvature, CZ a perpendicular upon it, $CO = r'$, $CZ = p'$,

$$PO = r\frac{dr}{dp}. \quad \text{(*Lacroix*, Elem. Treatise, Note H, p. 668.)}$$

$$p = \frac{e\sqrt{(a^2-r^2)}}{\sqrt{(a^2-e^2)}}; \therefore \frac{dp}{dr} = -\frac{er}{\sqrt{\{(a^2-e^2)(a^2-r^2)\}}},$$

$$PO = \frac{\sqrt{\{(a^2-e^2)(a^2-r^2)\}}}{e}.$$

$$\text{Also, } OZ = OP + CY = OP + p = \frac{a^2\sqrt{(a^2-r^2)}}{e\sqrt{(a^2-e^2)}},$$

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* *Principia*, Book I, Prop. 51.

$$CZ = \sqrt{(r^2 - p^2)} = \frac{a\sqrt{(r^2 - e^2)}}{\sqrt{(a^2 - e^2)}}.$$

But $OZ^2 = CO^2 - CZ^2$; whence $\dfrac{a^4(a^2 - r^2)}{e^2(a^2 - e^2)} = r'^2 - p'^2$,

and $p'^2 = r^2 - p^2$; whence $\dfrac{a^2(r^2 - e^2)}{(a^2 - e^2)} = p'^2$.

Multiply the first by e^2, and the second by a^2, and add, and we have, reducing,

$$a^4 = e^2 r'^2 - e^2 p'^2 + a^2 p'^2;$$

$$\therefore\ p'^2 = \frac{a^4 - e^2 r'^2}{a^2 - e^2}.$$

Let $\dfrac{a^2}{e} = a'$; whence $\dfrac{a^2}{e^2} = \dfrac{a'^2}{a^2}$,

then $p'^2 = \dfrac{a'^2 - r'^2}{\dfrac{a'^2}{a^2} - 1} = \dfrac{a^2(a'^2 - r'^2)}{a'^2 - a^2}$;

which is the same as the equation which we had to the hypocycloid, only putting a for e, and a' for a.

Hence, take $CS = \dfrac{a^2}{e}$, a third proportional to CA and CE; and describing a circle with radius CS, suppose a wheel, whose diameter is ES, to roll on this circle, so as to describe the hypocycloids, SD, Sd; these will be the evolutes of AD, Ad. And a pendulum oscillating between them will always have its extremity in the hypocycloid DAd.

Cor. 6. *The length of the hypocycloid* is thus found.*

$$\frac{ds}{dr} = \frac{r\sqrt{(a^2 - e^2)}}{a\sqrt{(r^2 - e^2)}};$$

$$\therefore\ s = \frac{\sqrt{\{(a^2 - e^2)(r^2 - e^2)\}}}{a},$$ measuring from A.

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* *Principia*, Book I, Props. 48, 49.

And for the whole arc AD, $s = \frac{a^2 - e^2}{a}$; since $r = a$.

Similarly, $SD = Sd = \frac{a'^2 - a^2}{a'} = \frac{a^4 - a^2 e^2}{a^2 e} = \frac{a^2 - e^2}{e}$.

Cor. 7. If the length of the pendulum $= l$, $\frac{a^2 - e^2}{e} = l$,

$$\text{whence, } a^2 - e^2 = le = \frac{la^2}{a'}.$$

And by Cor. 3, time of oscillation $= \pi \frac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} = \frac{\pi\sqrt{l}}{\sqrt{(ma')}}$.

Cor. 8. By Chap. I, Ex. 1, the time of falling to the centre C will be $\frac{\pi}{2\sqrt{m}} = \frac{\pi\sqrt{a'}}{2\sqrt{(ma')}}$. Hence, ½ time of an oscillation : time to centre :: $\sqrt{l} : \sqrt{a'} :: \sqrt{SA} : \sqrt{SC}$*.

If the curve be an *Epicycloid*, and the body be acted upon by a repulsive force, varying as the distance from the centre of the globe, we shall obtain similar results: and the oscillations will be isochronous†.

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* *Principia*, Book I, Prop. 52.

† If we resolve the force in these isochronous cases, so as to obtain the portion which acts along the curve, we shall find, that this portion is as the length of the curve from the lowest point; consequently, the force which accelerates the body, is as the space to be described; and, therefore, the time to the lowest point is independent of the original distance by Chap. I, p. 14.

Conversely, in any curve the descents to a given point will be isochronous, if the force be such, that the resolved part of it along the curve is as the arc from that point.

If the force be P, and act parallel to x, $P\frac{dx}{ds}$ will be the resolved part.

Ex. To find the force which must act in parallel lines, that the descents in a circle to the lowest point may be isochronous.

If c be the radius, and θ the angle which the radius CQ, fig. 3, makes with the vertical, x will be $= c$ ver. sin. θ, and $s = c\theta$; $\therefore \frac{dx}{ds} = \sin.\theta$. Hence, $P \sin.\theta$ must vary as $c\theta$, and P must vary as $\frac{\theta}{\sin.\theta}$, in order that the circle may be isochronous.

This problem is in the *Principia*, Book I, Prop. 53.

145. **Prop.** *When a body moves upon a curve, to find the magnitude of the reaction* R.

The reaction is also equal to the pressure upon the curve.

We resume the equations of Art. 143,

$$\frac{d^2x}{dt^2} = X + R\frac{dy}{ds};$$

$$\frac{d^2y}{dt^2} = Y - R\frac{dx}{ds}.$$

Multiplying the first by $\frac{dy}{dt}$, and the second by $\frac{dx}{dt}$, and subtracting, we have

$$\frac{dy}{dt}\,\frac{d^2x}{dt^2} - \frac{dx}{dt}\,\frac{d^2y}{dt^2} = X\frac{dy}{dt} - Y\frac{dx}{dt} + R\frac{ds}{dt},$$

because $\frac{dy}{ds}\,\frac{dy}{dt} + \frac{dx}{ds}\,\frac{dx}{dt} = \frac{dy^2}{ds^2}\,\frac{ds}{dt} + \frac{dx^2}{ds^2}\,\frac{ds}{dt} = \frac{ds}{dt}$.

Hence we have

$$R = -X\frac{dy}{dt}\,\frac{dt}{ds} + Y\frac{dx}{dt}\,\frac{dt}{ds} + \left(\frac{dy}{dt}\,\frac{d^2x}{dt^2} - \frac{dx}{dt}\,\frac{d^2y}{dt^2}\right)\frac{dt}{ds}$$

$$= Y\frac{dx}{ds} - X\frac{dy}{ds} + \left(\frac{dy}{ds}\,\frac{d^2x}{ds^2} - \frac{dx}{ds}\,\frac{d^2y}{ds^2}\right)\frac{ds^2}{dt^2}.$$

But if ρ be the radius of curvature, we have

$$\frac{1}{\rho} = \frac{dy}{ds}\,\frac{d^2x}{ds^2} - \frac{dx}{ds}\,\frac{d^2y}{ds^2},$$

$$\text{hence, } R = Y\frac{dx}{ds} - X\frac{dy}{ds} + \frac{1}{\rho}\,\frac{ds^2}{dt^2}.$$

Of the two parts of which this expression consists, the first is the force which we obtain by supposing X and Y resolved perpendicular to a tangent to the curve. The second, $\frac{1}{\rho}\frac{ds^2}{dt^2}$, or $\frac{(\text{velocity})^2}{\rho}$, is the force which would retain the body in the curve if it were moving in a circle whose radius is ρ. This latter is called the *centrifugal force*, and arises entirely from the tendency of the body to go in a straight line, instead of in the curvilinear path in which it is compelled to move: it is greater as the curvature is greater.

If the body be acted on by no force, $R = \frac{1}{\rho}\frac{ds^2}{dt^2} = \frac{(\text{velocity})^2}{\rho}$.

In the case where the body is acted on by gravity only,

$$R = g\frac{dy}{ds} + \frac{1}{\rho}\frac{ds^2}{dt^2}.$$

In the case where the force P tends to a centre at the origin of co-ordinates,

$$Y\frac{dx}{dt} - X\frac{dy}{dt} = P\left(\frac{x}{r}\frac{dy}{dt} - \frac{y}{r}\frac{dx}{dt}\right) = Pr\frac{dv}{dt};$$

for $x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{dv}{dt}$, (see Art. 22.)

$$\text{Hence } R = Pr\frac{dv}{ds} + \frac{1}{\rho}\frac{ds^2}{dt^2}.$$

Prob. IV. *A body oscillates in a cycloid acted on by gravity: to find the tension of the string.* Fig. 3.

$$R = g.\frac{dy}{ds} + \frac{1}{\rho}\frac{ds^2}{dt^2},$$

we have $\frac{dy}{dx} = \sqrt{\frac{2a-x}{x}}$, $\frac{ds}{dx} = \sqrt{\frac{2a}{x}}$, $\frac{dy}{ds} = \sqrt{\frac{2a-x}{2a}}$,

$\rho = 2\sqrt{\{2a(2a-x)\}}$, (*Lacroix*, Art. 103), $\frac{ds^2}{dt^2} = 2g(h-x)$.

$$\text{Hence, } R = g\sqrt{\frac{2a-x}{2a}} + \frac{g(h-x)}{\sqrt{\{2a.(2a-x)\}}}$$

$$= g.\frac{2a+h-2x}{\sqrt{(4a^2-2ax)}}.$$

When $x = h$, $R = g\dfrac{2a-h}{\sqrt{(4a^2-2ah)}} = g\dfrac{\sqrt{(2a-h)}}{\sqrt{(2a)}}$.

When $x = 0$, $R = g\dfrac{2a+h}{2a} = g\left(1+\dfrac{h}{2a}\right)$;

the part g arises from gravity, the other part $\dfrac{gh}{2a}$ from centrifugal force.

If $h = 2a$, or the body fall from the highest point; pressure at $A = 2g$.

Prob. V. *A body oscillates in a circular arc acted on by gravity: to find the tension of the string.*

$$\frac{dy}{dx} = \frac{c-x}{\sqrt{(2cx-x^2)}},\quad \frac{ds}{dx} = \frac{c}{\sqrt{(2cx-x^2)}},\quad \frac{dy}{ds} = \frac{c-x}{c},$$

$$\rho = c,\quad \frac{ds^2}{dt^2} = 2g(h-x);$$

$$\therefore\ R = g.\frac{c-x}{c} + \frac{2g(h-x)}{c} = g.\frac{c+2h-3x}{c}.$$

When $x = 0$, this gives $R = g\dfrac{c+2h}{c}$. If $h = c$, or the body fall through the whole quadrant, $R = 3g$; and the tension at the lowest point is three times the weight.

If the body fall through the whole semicircle from the highest point of the circle, $h = 2c$, $R = 5g$, and the tension at the lowest point is five times the weight.

This tension, by the increase of x, may become 0, and afterwards negative, or opposite to its former direction: the body will then tend to approach the centre of suspension.

It will $= 0$, when $c + 2h - 3x = 0$, or $x = \dfrac{c+2h}{3}$.

If a body move on the upper *convex surface* of a circle, it will only remain upon it while its pressure is towards the centre. It will fly off and describe a parabola, when the pressure becomes $= 0$; that is, when $x = \frac{c + 2h}{3}$; x and h being measured from the lowest point of the circle.

The parabola which the body describes, will be one which has a common tangent with the circle at the point where the body leaves it, and the velocity at that point

$$= \sqrt{\{2g \cdot (h - x)\}} = \sqrt{\left\{2g \cdot \frac{h - c}{3}\right\}}.$$

Since x is less than $2c$, we have $c + 2h < 6c$, or $h < \frac{5}{2} c$.

If h be greater than this, the body will not move along the circle at all, but will leave it at the highest point.

By a similar application of our formulæ we might easily find the tension, when a body oscillates in a *hypocycloid.*

Sect. II. THE MOTION OF A BODY ON A SURFACE OF REVOLUTION.

146. PROP. *To find the motion of a body upon a surface of revolution, and acted on by forces in a plane passing through the axis.*

Let CP, fig. 5, be the curve, by the revolution of which the surface is described; AC its axis: and let AMN be a fixed plane perpendicular to this axis, AM, MN, NP three rectangular co-ordinates to the point P; represented by x, y, z, respectively. Also let OLP be a plane parallel to AMN; and let $OP = r$. Then since $AO = NP = z$, and $OP = r$, knowing the nature of the curve CP, we know the relation of r and z.

The forces which act upon the body when it is at P, are the reaction of the surface and the extraneous force: let this latter, which is by supposition in the plane POA, be resolved into two, P in the direction PO, and Z in the direction PN. Also the reaction is manifestly perpendicular to the curve CP, and in the plane AOP; and if we resolve it, as in the case of a plane curve, we shall have its component

$$\text{in direction } PO = R\frac{dz}{ds},$$

$$\text{in direction } NP = R\frac{dr}{ds};$$

supposing $\dfrac{ds}{dr} = \sqrt{\left(1 + \dfrac{dz^2}{dr^2}\right)}$.

Hence, the whole force in $PO = P + R\dfrac{dz}{ds}$, and if we resolve this force in directions parallel to x and y, we shall have, since OL, LP, OP are x, y, r, respectively,

$$\text{component in } x = -\left(P + R\frac{dz}{ds}\right)\cdot\frac{x}{r},$$

$$\text{in } y = -\left(P + R\frac{dz}{ds}\right)\cdot\frac{y}{r}.$$

$$\text{Also force in } z = -\ \ Z + R\frac{dr}{ds}.$$

Hence, by the equations (a') we have

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} &= -\left(P + R\frac{dz}{ds}\right)\cdot\frac{x}{r},\\
\frac{d^2y}{dt^2} &= -\left(P + R\frac{dz}{ds}\right)\cdot\frac{y}{r},\\
\frac{d^2z}{dt^2} &= -Z + R\frac{dr}{ds}.
\end{aligned}\right\}\ldots\ldots\ldots(1).$$

Hence, we find

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;\ \text{ or }\ \frac{d\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right)}{dt} = 0\ldots\ldots(2).$$

Also,

$$\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{dy}{dt}\frac{d^2y}{dt^2}+\frac{dz}{dt}\frac{d^2z}{dt^2}=-P\left(\frac{x}{r}\frac{dx}{dt}+\frac{y}{r}\frac{dy}{dt}\right)-Z\frac{dz}{dt}$$

$$-R\frac{dz}{ds}\left(\frac{x}{r}\frac{dx}{dt}+\frac{y}{r}\frac{dy}{dt}-\frac{dr}{dt}\right),$$

because $\frac{dr}{ds}\frac{dz}{dt}=\frac{dz}{ds}\frac{dr}{dt}$.

But the coefficient of R is 0, for since $x^2+y^2=r^2$, we have

$$\frac{x}{r}\frac{dx}{dt}+\frac{y}{r}\frac{dy}{dt}=\frac{dr}{dt}.$$

Hence substituting this, and integrating with respect to t,

$$\tfrac{1}{2}\left(\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)=-\int_t P\frac{dr}{dt}-\int_t Z\frac{dz}{dt};$$

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}=-2\int_r P-2\int_z Z\ldots\ldots(3).$$

Now $\frac{dz^2}{dt^2}=\frac{dz^2}{dr^2}\cdot\frac{dr^2}{dt^2}$; and from the nature of the curve CP, $\frac{dz}{dr}$ is known in terms of r. Let $\frac{dz}{dr}=p$, p representing the tangent of the angle which the surface at P makes with the horizon: therefore $\frac{dz^2}{dt^2}=p^2\cdot\frac{dr^2}{dt^2}$.

Also let the angle $LOP=v$, and we shall have, (as in Art. 22, 23,)

$$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{dv}{dt}$$

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}=\frac{dr^2}{dt^2}+r^2\frac{dv^2}{dt^2}$$

Hence by (2) and (3) $\frac{d \, . \, \frac{r^2 dv}{dt}}{dt} = 0,$

$$\frac{dr^2}{dt^2} + \frac{r^2 dv^2}{dt^2} + p^2 \, . \, \frac{dr^2}{dt^2} = -2 \int_r P - 2 \int_z Z.$$

If we integrate the first of these, we obtain

$$r^2 \frac{dv}{dt} = h \text{ ; } h \text{ being a constant quantity,}$$

$$\text{or } \frac{dt}{dv} = \frac{r^2}{h}.$$

The second of the two equations just found, might be integrated, if we could perform the integrations on the right-hand side $-2\int_r P - 2\int_z Z$. If we put for P, Z, and z, their values in terms of r, this expression will become a function of r, and its integral will be a function of r, for $\int_z Z = \int_r Z \frac{dz}{dr}$.

Let $\int_z \left(P + Z\frac{dz}{dr}\right) = Q$: hence

$$\frac{dr^2}{dt^2} + r^2 \frac{dv^2}{dt^2} + p^2 \, . \, \frac{dr^2}{dt^2} = C - 2Q.$$

To find the path of the body, put $\frac{dr}{dv}\frac{dv}{dt}$ for $\frac{dr}{dt}$, and for $\frac{dv}{dt}$ its value $\frac{h}{r^2}$, and we have

$$(1 + p^2) \, . \, \frac{h^2}{r^4} \frac{dr^2}{dv^2} + \frac{h^2}{r^2} = C - 2Q;$$

$$\therefore \frac{dv}{dr} = \frac{\sqrt{(1 + p^2)} \, . \, h}{r\sqrt{\{(C - 2Q) r^2 - h^2\}}} \text{ ....} (e).$$

$$\text{Hence, } \frac{dt}{dr} = \frac{\sqrt{(1 + p^2)} \, . \, r}{\sqrt{\{(C - 2Q) r^2 - h^2\}}} \text{ ....} (f).$$

If equation (*e*) be integrable, we have v in terms of r, whence the locus of N is known. And r being known, we know z; for $z = \int_r p$.

If the force be always parallel to the axis, we have $P = 0$; and if also Z be a constant force, as gravity, and $= g$, $Q = \int_z z = gz$, and z is to be expressed in terms of r.

If the force tend to a fixed point in the axis, we may make this point the origin A. Let $AP = r'$, and the force in $PA = P'$; therefore $P = P'\frac{r}{r'}$, $Z = P'\frac{z}{r'}$; and $Q = \int_r P + \int_z Z$

$$= \int_{r'}\left(P'\frac{r}{r'}\frac{dr}{dr'} + P'\frac{z}{r'}\frac{dz}{dr'}\right) = \int_{r'} P',$$

because $r^2 + z^2 = r'^2$. And if P' be a function of r' we find Q by integrating.

There will be apsides when $\frac{dr}{dv} = 0$, and therefore when

$$(C - 2Q)\,r^2 - h^2 = 0.$$

147. Prop. *To find under what circumstances a body will describe a circle on a surface of revolution.*

For this purpose it must always move in a plane perpendicular to the axis of revolution; r and z will be constant; and hence, by last Article $\frac{dv}{dt}$ is constant. Also $r\cos. v = x$; therefore by differentiation, $\frac{d^2x}{dt^2} = -r\cos. v\frac{dv^2}{dt^2}$; and if V be the velocity, $V = r\frac{dv}{dt}$; $\therefore \frac{d^2x}{dt^2} = -\frac{V^2\cos. v}{r}$.

Hence, the first and third of equations (1) in last Article become

$$\frac{V^2}{r} = P + R\frac{dz}{ds},$$

$$0 = -Z + R\frac{dr}{ds};$$

$$\therefore \frac{V^2}{r} = P + Z\frac{dz}{dr}\text{.........}(g).$$

If the force be gravity acting in the direction of the axis *OA*, so that $P = 0$, $Z = g$; $\dfrac{V^2}{r} = g\dfrac{dz}{dr}$.

Since $h = 2$ area in time $1 = Vr$, $h^2 = V^2r^2 = gr^3\dfrac{dz}{dr} = gr^3p$.

Prop. VI. *To determine the time of a pendulum performing a conical revolution.*

A body suspended by a string *SP* from a point *S*, fig. 6, will be retained in a spherical surface whose centre is *S*, and may, by properly adjusting the velocity, revolve in a circle, *SP* describing a conical surface, having a vertical axis *SC*, which is the motion here spoken of.

In a spherical surface of which the radius is c,

$$z = AS - \sqrt{(c^2 - r^2)}\,; \quad \frac{dz}{dr} = \frac{r}{\sqrt{(c^2 - r^2)}},$$

$$V^2 = \frac{gr^2}{\sqrt{(c^2 - r^2)}}, \quad V = \frac{r\sqrt{g}}{(c^2 - r^2)^{\frac{1}{4}}}.$$

If we draw *PT* a tangent at *P*, meeting *SC*,

$$OT = \frac{r^2}{\sqrt{(c^2 - r^2)}}\,; \quad V^2 = 2g\,.\,\tfrac{1}{2}OT\,;$$

hence, the velocity of *P* is that acquired by falling down half *OT*.

The circle described by *P* has its circumference $= 2\pi r$; hence, we have the time of a revolution

$$= \frac{2\pi r}{V} = \frac{2\pi(c^2 - r^2)^{\frac{1}{4}}}{\sqrt{g}} = \frac{2\pi\sqrt{SO}}{\sqrt{g}}.$$

Cor. 1. By Prob. I, Cor. 3, the time of a double oscillation of a pendulum whose length is l, would be $2\pi\sqrt{\dfrac{l}{g}}$. Hence, in order that this oscillation may employ the same time as the revolution, we must have $l = SO$.

Cor. 2. The last corollary is true for any surface, PS being a normal.

Cor. 3. If a body were to revolve in a parabolical surface, the times of all circular revolutions would be the same. For in this case, SO the subnormal is constant.

Prob. VII. *A body moves in a spherical surface acted on by gravity, and so as not to describe a circle: to determine its motion.*

Let c be the radius of the sphere. We have $Q = gx$, and $C - 2Q = 2g(k - x)$, k being an arbitrary quantity.

Also $r^2 = 2cx - x^2$, x being measured from the surface:

$$r\frac{dr}{dx} = (c - x)\,; \quad 1 + p^2 = 1 + \frac{r^2}{(c-x)^2} = \frac{c^2}{(c-x)^2}.$$

Hence, by equation (f) $\dfrac{dt}{dr} = \dfrac{\sqrt{(1+p^2)}\,r}{\sqrt{\{(C - 2Q)\,r^2 - h^2\}}}$

$$\frac{dt}{dx} = \frac{c}{\sqrt{\{2g(k-x)(2cx-x^2) - h^2\}}}.$$

In order that $\dfrac{dx}{dt}$ may $= 0$, the denominator of the right-hand member of this equation must $= 0$; that is,

$$2g(k - x)(2cx - x^2) - h^2 = 0\,;$$

$$\text{or } x^3 - (k + 2c)\,x^2 + 2kcx - \frac{h^2}{2g} = 0\,;$$

which equation will necessarily have two possible roots; because, as the body moves, it will necessarily reach one highest and one lowest point, and therefore two places where $\dfrac{dx}{dt} = 0$. Hence, the equation has also a third possible root. Suppose it to be identical with

$$(x - \alpha)(x - \beta)(x - \gamma) = 0:$$

where α is the greatest value of x, and β the least, which occur during the body's motion.

Hence, $\dfrac{dt}{dx} = \dfrac{c}{\sqrt{(2g)} \cdot \sqrt{\{(a-x)(x-\beta)(\gamma-x)\}}}$.

To integrate this, let $\theta = \text{arc}\left(\sin. = \sqrt{\dfrac{x-\beta}{a-\beta}}\right)$;

$$\therefore \frac{d\theta}{dx} = \frac{1}{2\sqrt{\{(x-\beta)(a-\beta)\}} \cdot \sqrt{\left\{1 - \dfrac{x-\beta}{a-\beta}\right\}}}$$

$$= \frac{1}{2\sqrt{\{(a-x)(x-\beta)\}}}.$$

Also $\sin.^2\theta = \dfrac{x-\beta}{a-\beta}$; $\therefore\ x = \beta + (a-\beta)\sin.^2\theta$.

And $\gamma - x = \gamma - \{\beta + (a-\beta)\sin.^2\theta\} = (\gamma-\beta)\{1 - \delta^2\sin.^2\theta\}$, if $\delta = \sqrt{\dfrac{a-\beta}{\gamma-\beta}}$;

$$\therefore \frac{dt}{d\theta} = \frac{2c}{\sqrt{\{2g(\gamma-\beta)\}} \cdot \sqrt{\{1-\delta^2\sin.^2\theta\}}},$$

to be integrated from $x = \beta$ to $x = a$; that is, from $\theta = 0$, to $\theta = \dfrac{\pi}{2}$: this expanded in the same manner as in Prob. II, gives

$$t = \frac{2c}{\sqrt{2g(\gamma-\beta)}}\left\{1 + \left(\frac{\delta}{2}\right)^2 + \left(\frac{1.3.\delta^2}{2.4}\right)^2 \right.$$

$$\left. + \left(\frac{1.3.5.\delta^3}{2.4.6}\right)^2 + \&c.\right\}.\frac{\pi}{2};$$

which is the time of a whole oscillation from the least to the greatest distance.

Also $\dfrac{dv}{dt} = \dfrac{h}{r^2} = \dfrac{h}{2cx - x^2}$, and is hence known in terms of x.

148. **Prop.** *A body acted on by gravity moves on a surface of revolution, whose axis is vertical: when the body's path is nearly circular, it is required to find the angle between the apsides of* N's *path, fig.* 5.

In this case, $\int_z Z = gz = Q$.

And if at an apse $r = a$, $z = k$, we have

$$(C - 2gk)\,a^2 - h^2 = 0; \quad \therefore \; C = \frac{h^2}{a^2} + 2gk.$$

Hence, equation (e), Art. 146, becomes

$$\frac{dv}{dr} = \frac{\sqrt{(1 + p^2)}\,h}{r\sqrt{\left\{2g\,(k - z)\,r^2 - h^2\dfrac{(a^2 - r^2)}{a^2}\right\}}}$$

$$= \frac{\sqrt{(1 + p^2)} \cdot \dfrac{h}{r^2}}{\sqrt{\left\{2g\,(k - z) - h^2\left(\dfrac{1}{r^2} - \dfrac{1}{a^2}\right)\right\}}}.$$

$$\text{Let } \frac{1}{r} = \frac{1}{a} + w; \quad \therefore \; \frac{dr}{dw} = -r^2,$$

$$\frac{dv}{dw} = -\frac{\sqrt{(1 + p^2)} \cdot h}{\sqrt{\left\{2g\,(k - z) - h^2\left(\dfrac{1}{r^2} - \dfrac{1}{a^2}\right)\right\}}}.$$

$$\text{Hence } \frac{dw^2}{dv^2} = \frac{1}{1 + p^2}\left\{\frac{2g}{h^2}(k - z) - \left(\frac{1}{r^2} - \frac{1}{a^2}\right)\right\}.$$

It is requisite to express the right-hand side of this equation in terms of w.

Now since at the apse we have $w = 0$, $z = k$,

$$\text{we have generally } z = k + \frac{dz}{dw} \cdot w + \frac{d^2z}{dw^2} \cdot \frac{w^2}{2} + \&\text{c.}$$

the values of the differential coefficients being taken for $w = 0$.

And $\dfrac{dz}{dw} = \dfrac{dz}{dr} \cdot \dfrac{dr}{dw} = -pr^2$, because $\dfrac{dr}{dw} = -r^2$,

$$\frac{d^2z}{dw^2} = -\frac{d \, . \, pr^2}{dw} = -2pr\frac{dr}{dw} - r^2\frac{dp}{dw};$$

$$= -(2pr + qr^2)\frac{dr}{dw}, \quad \text{if } \frac{dp}{dr} = q$$

$$= (2p + qr)\,r^3.$$

If p_1 and q_1 be the values which p and q assume when $w = 0$, and $r = a$, we have for that case

$$\frac{dz}{dw} = -p_1 a^2, \quad \frac{d^2z}{dw^2} = (2p_1 + q_1 a)\,a^3. \quad \text{Whence}$$

$$z = k - p_1 a^2 w + (2p_1 + q_1 a)\frac{a^3 w^2}{2} + \&\text{c.}$$

When a body moves in a circle of radius $= a$, we have $h^2 = gr^3p$ and $= ga^3p$, in this case (Art. 147.). And when the body moves *nearly* in a circle, h^2 will have nearly this value. Let

$$h^2 = \frac{ga^3p_1}{1+\delta}, \quad \delta \text{ being small. Then } \frac{2g}{h^2}(k - z) \text{ becomes}$$

$$(1+\delta)\left\{\frac{2w}{a} - \left(2 + \frac{q_1 a}{p_1}\right)aw^2 + \&\text{c.}\right\}.$$

$$\text{Also } \frac{1}{r^2} = \left(\frac{1}{a} + w\right)^2; \quad \frac{1}{r^2} - \frac{1}{a^2} = \frac{2w}{a} + w^2.$$

Hence $\dfrac{2g}{h^2}(k - z) - \left(\dfrac{1}{r^2} - \dfrac{1}{a^2}\right)$ becomes

$$\frac{2\delta w}{a} - \left(3 + \frac{q_1 a}{p_1}\right)w^2 + \&\text{c.}$$

Also $p = p_1 + Aw + \&\text{c.}$, whence $\dfrac{1}{1+p^2} = \dfrac{1}{1+p_1^2} + Bw + \&\text{c.}$

Hence $$\frac{dw^2}{dv^2} = \frac{1}{1+p_1^2}\left\{\frac{2\delta w}{a} - \left(3 + \frac{q_1 a}{p_1}\right) w^2\right\},$$
the higher powers of w being neglected in comparison of w^2.

Differentiate and divide by $2\dfrac{dw}{dv}$, and we have
$$\frac{d^2w}{dv^2} = \frac{1}{1+p_1^2}\left\{\frac{\delta}{a} - \left(3 + \frac{q_1 a}{p_1}\right) w\right\}, \text{ or}$$
$$\frac{d^2w}{dv^2} = N(\alpha - w),$$
$$\text{if } N = \frac{3p_1 + q_1 a}{p_1(1+p_1^2)}, \quad \alpha = \frac{p_1\delta}{a(3p_1 + q_1 a)}.$$

The integral of this equation, taken so that when $v = 0$, $w = \alpha$, is
$$\alpha - w = C \sin.\, v\sqrt{N}.$$

Now w passes from its least to its greatest value, and consequently r passes from its maximum to its minimum, while the arc $v\sqrt{N}$ passes from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$. Hence, for the angle A between the apsides, we have
$$A\sqrt{N} = \pi, \quad A = \frac{\pi}{\sqrt{N}} = \pi\sqrt{\frac{p_1(1+p_1^2)}{3p_1 + q_1 a}}.$$
$$\text{Also } \frac{dw}{dv} = -C \cos.\, v\sqrt{N};$$

and comparing this with the value of $\dfrac{dw^2}{dv^2}$, it appears that $C = \alpha$. Hence
$$w = \alpha\{1 - \sin.\, v\sqrt{N}\},$$
and the limits of the variation of w are from 0 to 2α, which is
$$\frac{2p_1\delta}{a(3p_1 + q_1 a)}.$$

Also $r = a - a^2 w$, neglecting the higher powers; and the limits of r's variation are from a to $a - \frac{2p_1 a\delta}{3p_1 + q_1 a}$; where δ is given from the apsidal velocity, and p_1 and q_1 from the form of the surface.

Prob. VIII. *Let the surface be a sphere; and let the path described be nearly a circle: to find the horizontal angle between the apsides.*

Supposing the origin to be at the lowest point of the surface, we have

$$x = c - \sqrt{(c^2 - r^2)}\,;\ p = \frac{dx}{dr} = \frac{r}{\sqrt{(c^2 - r^2)}}\,;\ q = \frac{dp}{dr} = \frac{c^2}{(c^2 - r^2)^{\frac{3}{2}}}\,;$$

$$\therefore\ p_1 = \frac{a}{\sqrt{(c^2 - a^2)}}\,;\ q_1 = \frac{c^2}{(c^2 - a^2)^{\frac{3}{2}}}\,;\ 1 + p_1^2 = \frac{c^2}{c^2 - a^2}\,;$$

$$\therefore\ N = \frac{4c^2 - 3a^2}{c^2}.$$

Hence, the angle between the apsides $= \frac{\pi c}{\sqrt{(4c^2 - 3a^2)}}$.

The motion of a point on a spherical surface, is manifestly the same as the motion of a simple pendulum or heavy body, suspended by an inextensible string from a fixed point; the body being considered as a point, and the string without weight. If the pendulum begin to move in a vertical plane, it will go on oscillating in the same plane, in the manner already considered in Sect. I, of this Chapter. But, if the pendulum have any lateral motion, it will go on revolving about the lowest point, and generally alternately approaching to it, and receding from it. By a proper adjustment of the velocity and direction, it may describe, by Art. 147, a circle; and if the velocity, when it is moving parallel to the horizon, be *nearly* equal to the velocity in a circle, it will describe a curve little differing from a circle. In this case, we can find the angle between the greatest and least distances, by the formula just deduced.

$$\text{Since } A = \frac{\pi c}{\sqrt{(4c^2 - 3a^2)}};$$

If $a = 0$, $A = \frac{\pi}{2}$; the apsides are 90^0 from each other, which also appears by observing, that when the amplitude of the pendulum's revolutions is very small, the force is nearly as the distance; and the body describes ellipses nearly; of which the lowest point is the centre.

If $a = c$, $A = \pi = 180^0$; this is when the pendulum string is horizontal; and requires an infinite velocity.

If $a = \frac{c}{2}$; so that the string is inclined 30^0 to the vertical;

$$A = \frac{2\pi}{\sqrt{13}} = 99^0\,50'.$$

If $a^2 = \frac{c^2}{2}$; so that the string is inclined 45^0 to the vertical;

$$A = \pi\sqrt{\frac{2}{5}} = 113^0.\,56'.$$

If $a^2 = \frac{3c^2}{4}$; so that the string is inclined 60^0 to the vertical;

$$A = \frac{2\pi}{\sqrt{7}} = 136^0, \text{ nearly.}$$

PROB. IX. *Let the surface be an inverted cone, with its axis vertical: to find the horizontal angle between the apsides when the orbit is nearly a circle.*

Let a be the radius of the circle, and γ the angle which the slant side makes with the horizon.

Hence, $z = r\tan.\gamma$, $p = \tan.\gamma$, $q = 0$;

$$\therefore \text{ by p. 29, } N = \frac{3\tan.\gamma}{\tan.\gamma.\sec.^2\gamma} = 3\cos.^2\gamma:$$

and the angle between the apsides $= \frac{\pi}{\cos.\gamma.\sqrt{3}}$.

If $\gamma = 60^\circ$, angle $= \frac{2\pi}{3} = 120^\circ$.

Prob. X. *Let the surface be an inverted paraboloid whose parameter is* c.

$$r^2 = cx; \quad \therefore\ p = \frac{dx}{dr} = \frac{2r}{c}; \quad q = \frac{2}{c};$$

$$\therefore\ N = \frac{\dfrac{6a}{c} + \dfrac{2a}{c}}{\dfrac{2a}{c}\left(1 + \dfrac{4a^2}{c}\right)} = \frac{4c^2}{c^2 + 4a^2}.$$

If $a = \frac{c}{2}$, or the body revolve at the extremity of the focal ordinate,

$$N = 2; \text{ angle between apsides} = \frac{\pi}{\sqrt{2}}.$$

Prob. XI. *Let in fig.* 7, OP *vary inversely as* CO^3.

$$\text{Let } AC = h,\ h - x = \frac{c^4}{r^3}; \quad \therefore\ p = \frac{dx}{dr} = \frac{3c^4}{r^4}; \quad q = -\frac{12c^4}{r^5};$$

$$\therefore\ N = \frac{\dfrac{9c^4}{a^4} - \dfrac{12c^4}{a^4}}{\dfrac{3c^4}{a^4}\left(1 + \dfrac{9c^8}{a^8}\right)} = -\frac{a^8}{a^8 + 9c^8}.$$

This is negative, and hence the angle is impossible, and the body will never come to a second apse. If the velocity be at all less than that in a circle, the body will go on descending continually down the funnel *PA*.

149. **Prop.** *When a body moves on a conical surface, acted on by a force tending to the vertex; its motion in the surface will be the same, as if the surface were unwrapped, and made plane, the force remaining at the vertex.*

Let A, fig. 5, be at the vertex, $AP = r'$, force $= P'$, and the angle which the slant side makes with the base $= \gamma$; $\therefore z = r \tan.\gamma$, $p = \tan.\gamma$, $1 + p^2 = \sec.^2\gamma$.

Also $Q = \int_r P + \int_z Z = \int_{r'} P'$.

Hence, by equation (e), Art. 146,

$$\frac{dv}{dr} = \frac{\sec.\gamma \,.\, h}{r\sqrt{\{(C - 2\int_{r'} P')r^2 - h^2\}}}.$$

Now $\dfrac{dv}{dr} = \dfrac{dv}{dv'}\,\dfrac{dv'}{dr'}\,\dfrac{dr'}{dr}$: also $r = r'\cos\gamma$, whence $\dfrac{dr'}{dr} = \dfrac{1}{\cos.\gamma}$.

Assume $\dfrac{dv'}{dv} = \sec.\gamma$ and $h' = h\sec.\gamma$. Then

$$\frac{dv'}{dr'} = \frac{h'}{r'\sqrt{\{(C - 2\int_{r'} P')r'^2 - h'^2\}}}.$$

Now $\dfrac{dv'}{dr'}$ is the differential coefficient of the angle described *along the conical surface*, and it appears that the relation between v' and r' will be the same as it would be in a plane, where a body was acted upon by a central force P'. For we have in that case, (see Part I.)

$$\frac{h'^2}{r'^4}\,\frac{dr'^2}{dv'^2} + \frac{h'^2}{r'^2} = C - 2\int_{r'} P',$$

which agrees with the equation just found.

150. Prop. *When a body moves on a surface of revolution, to find the reaction* R.

Take the three original equations, (1) p. 20, and multiply them respectively by $x\dfrac{dz}{dt}$, $y\dfrac{dz}{dt}$ and $r\dfrac{dr}{dt}$; and the two first become

$$x\frac{dz}{dt}\,\frac{d^2x}{dt^2} = -P\frac{x^2}{r}\,\frac{dz}{dt} - R\frac{x^2}{r}\,\frac{dz}{ds}\,\frac{dz}{dt},$$

$$y\frac{dz}{dt}\,\frac{d^2y}{dt^2} = -P\frac{y^2}{r}\,\frac{dz}{dt} - R\frac{y^2}{r}\,\frac{dz}{ds}\,\frac{dz}{dt}.$$

Add these, observing that $x^2+y^2=r^2$, and we have

$$\frac{dz}{dt}\left(x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}\right)=-Pr\frac{dz}{dt}-Rr\frac{dz}{ds}\,\frac{dz}{dt}:$$

and the third is

$$r\frac{dr}{dt}\,\frac{d^2z}{dt^2}=-Zr\frac{dr}{dt}+Rr\frac{dr}{ds}\,\frac{dr}{dt}.$$

Subtract this, observing that

$$\frac{dz}{ds}\,\frac{dz}{dt}+\frac{dr}{ds}\,\frac{dr}{dt}=\left(\frac{dz^2}{ds^2}+\frac{dr^2}{ds^2}\right)\frac{ds}{dt}=\frac{ds}{dt};$$

and we have

$$\frac{dz}{dt}\left(x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}\right)-r\frac{dr}{dt}\,\frac{d^2z}{dt^2}=r\left(Z\frac{dr}{dt}-P\frac{dz}{dt}\right)-Rr\frac{ds}{dt}.$$

But $x^2+y^2=r^2$, $x\dfrac{dx}{dt}+y\dfrac{dy}{dt}=r\dfrac{dr}{dt}$

whence $x\dfrac{d^2x}{dt^2}+y\dfrac{d^2y}{dt^2}+\dfrac{dx^2}{dt^2}+\dfrac{dy^2}{dt^2}=r\dfrac{d^2r}{dt^2}+\dfrac{dr^2}{dt^2}$;

hence the equation becomes

$$\frac{dz}{dt}\left(\frac{dr^2}{dt^2}-\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}\right)+r\left(\frac{dz}{dt}\,\frac{d^2r}{dt^2}-\frac{dr}{dt}\,\frac{d^2z}{dt^2}\right)=r\left(Z\frac{dr}{dt}-P\frac{dz}{dt}\right)-Rr\frac{ds}{dt}.$$

Also, $\dfrac{dr^2}{dt^2}=\dfrac{ds^2}{dt^2}-\dfrac{dz^2}{dt^2}$; hence

$$R=Z\frac{dr}{ds}-P\frac{dz}{ds}+\frac{dr}{ds}\,\frac{d^2z}{dt^2}-\frac{dz}{ds}\,\frac{d^2r}{dt^2}+\frac{dz}{ds}\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}-\frac{ds^2}{dt^2}\right).$$

Now, if ρ be the radius of curvature of CP at P,

$$\frac{1}{\rho}=\left(\frac{dr}{ds}\,\frac{d^2z}{dt^2}-\frac{dz}{ds}\,\frac{d^2r}{dt^2}\right)\frac{dt^2}{ds^2},$$

t being any independent variable; also let σ be the arc described, whence

$$\frac{d\sigma^2}{dt^2}=\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}.$$

Hence $R = Z\dfrac{dr}{ds} - P\dfrac{dz}{ds} + \dfrac{1}{\rho}\dfrac{ds^2}{dt^2} + \dfrac{dz}{ds}\left(\dfrac{d\sigma^2}{dt^2} - \dfrac{ds^2}{dt^2}\right)$.

Here it is manifest that $\dfrac{ds^2}{dt^2}$ is the square of the velocity resolved in the curve *PC*, fig. 5, and that $\dfrac{d\sigma^2}{dt^2} - \dfrac{ds^2}{dt^2}$ is the square of the velocity resolved perpendicular to *OP* in the plane *OLP*.

The two last terms, which involve these quantities, together form that part of the resistance which is due to the centrifugal force; the first term is that part which arises from the resolved parts of the forces.

From this expression we know the value of R; for we have, as before,

$$\frac{d\sigma^2}{dt^2} = C - 2\left(\textstyle\int_r P + \int_z Z\right);$$

also $\dfrac{d\sigma^2}{dt^2} - \dfrac{ds^2}{dt^2} = \text{vel.}^2$ in $PU = r^2\dfrac{dv^2}{dt^2} = \dfrac{h^2}{r^2}$.

Hence, $\dfrac{ds^2}{dt^2} = C - 2\left(\int_r P + \int_z Z\right) - \dfrac{h^2}{r^2}$.

Prob. XII. *To find the tension of a pendulum moving in a spherical surface.*

Retaining the denominations of the Proposition, we have

$$\rho = 0; \;\; C - 2\left(\textstyle\int_r P + \int_z Z\right) = 2g(k - z),$$

$$r = \sqrt{(2cz - z^2)}; \;\; \frac{dr}{dz} = \frac{c - z}{\sqrt{(2cz - z^2)}}; \;\; \frac{ds}{dr} = \frac{c}{c - z};$$

$$\frac{ds}{dz} = \frac{c}{\sqrt{(2cz - z^2)}} = \frac{c}{r}.$$

Hence, $R = \frac{g(c-z)}{c} + \frac{2g(k-z) - \frac{h^2}{r^2}}{c} + \frac{h^2}{r^3}\cdot\frac{r}{c}$

$= \frac{g\cdot(c+2k-3z)}{c}$,

and hence it is the same as that of the pendulum oscillating in a vertical plane with the same velocity at the same distances: see Prob. V.

Sect. III. THE MOTION OF A POINT UPON ANY SURFACE.

151. PROP. *To find the velocity, reaction, and motion of a body upon any surface whatever.*

Let R be the reaction of the surface, which is of course in the direction of a normal to it at each point. And let ϵ, ϵ', ϵ'', be the angles which this normal makes with the axes of x, y, and z respectively; we shall then have, considering the resolved parts of R among the forces which act on the point, by the formulæ (c'),

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= X + R\cos.\epsilon,\\ \frac{d^2y}{dt^2} &= Y + R\cos.\epsilon',\\ \frac{d^2z}{dt^2} &= Z + R\cos.\epsilon''.\end{aligned}\right\}\text{............ (1).}$$

Now the nature of the surface is expressed by an equation between x, y, z: and if we suppose that we have, deduced from this equation,

$$\frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt},$$

t being any independent variable, and p, q, being the partial differential coefficients of z; (*Lacroix*, 125); we have for the equations to the normal of the point whose co-ordinates are

x, y, z, (*Lacroix*, Elem. Treat. No. 143); x', y', z', being co-ordinates to any point in the normal;

$$x' - x + p\,(z' - z) = 0,$$
$$y' - y + q\,(z' - z) = 0.$$

Hence, it appears that if PK, fig. 8, be the normal, PG, PH its projections on planes parallel to xz and yz respectively; the equation of PG is $x' - x + p\,(z' - z) = 0$; and hence $GN + p\,.\,PN = 0$, and $GN = -p\,.\,PN$, similarly the equation of PH is $y' - y + q\,(z' - z) = 0$, whence $HN + q\,.\,PN = 0$, and $HN = -q\,.\,PN$.

And hence, $\cos.\,\epsilon = \cos.\,KPh = \dfrac{Ph}{PK} = \dfrac{GN}{\sqrt{(PN^2 + NG^2 + HN^2)}}$

$$\text{or, } \cos.\,\epsilon = -\frac{p}{\sqrt{(1 + p^2 + q^2)}}$$

$$\cos.\,\epsilon' = \cos.\,KPg = \frac{Pg}{PK} = -\frac{HN}{\sqrt{(PN^2 + NG^2 + HN^2)}}$$

$$\text{or, } \cos.\,\epsilon' = -\frac{q}{\sqrt{(1 + p^2 + q^2)}}:$$

$$\text{whence, } \cos.\,\epsilon'' = \sqrt{(1 - \cos.^2\epsilon - \cos.^2\epsilon')} = \frac{1}{\sqrt{(1 + p^2 + q^2)}}.$$

Substituting these values; multiplying by

$$\frac{dx}{dt},\ \frac{dy}{dt},\ \text{and } \frac{dz}{dt}{}^{*}$$

respectively, in the three equations; and observing that

$$\frac{dz}{dt} - p\,\frac{dx}{dt} - q\,\frac{dy}{dt} = 0, \text{ we have}$$

$$\frac{dx}{dt}\,\frac{d^2x}{dt^2} + \frac{dy}{dt}\,\frac{d^2y}{dt^2} + \frac{dz}{dt}\,\frac{d^2z}{dt^2} = X\,\frac{dx}{dt} + Y\,\frac{dy}{dt} + Z\,\frac{dz}{dt}.$$

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[*] Here $\frac{dz}{dt}$ indicates the differential coefficient of z considered as a function of *both* the quantities x and y. It is what *Lacroix*, Elem. Treat. 126, would indicate by $\frac{d(z)}{dt}$.

Hence, $$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = 2\int_t \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right),$$

and if this can be integrated, we have the velocity.

If we take the three original equations (1), and multiply them respectively by $-p$, $-q$, and 1, and then add, we obtain

$$-p\frac{d^2x}{dt^2} - q\frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} = -pX - qY + Z + R\sqrt{(1+p^2+q^2)}.$$

But $$\frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt};$$

hence, $$\frac{d^2z}{dt^2} = p\frac{d^2x}{dt^2} + q\frac{d^2y}{dt^2} + \frac{dp}{dt}\frac{dx}{dt} + \frac{dq}{dt}\frac{dy}{dt};$$

substituting this on the first side of the above equation, and taking the value of R, we find

$$R = \frac{pX + qY - Z}{\sqrt{(1+p^2+q^2)}} + \left(\frac{dp}{dt}\frac{dx}{dt} + \frac{dq}{dt}\frac{dy}{dt}\right).$$

If in the three original equations we eliminate R, we find two second differential equations, involving the known forces X, Y, Z, and p, q, which are also known when the surface is known: combining with these the equation to the surface, by which z is known in x and y, we have equations from which we can find the relation between the time and the three co-ordinates.

152. Prop. *To find the path which a body will describe upon a given surface when acted upon by no force.*

In this case we must make X, Y, Z, each $= 0$.

We have here to eliminate t, and hence we must no longer take t for the independent variable. Let the equations (1) of Article 151, be thus written, omitting X, Y, Z.

$$\frac{d\frac{dx}{dt}}{dt} = R\cos.\epsilon, \quad \frac{d\frac{dy}{dt}}{dt} = R\cos.\epsilon', \quad \frac{d\frac{dz}{dt}}{dt} = R\cos.\epsilon''.$$

Multiply them respectively by

$$-(qw+v),\quad pw+u,\quad qu-pv,$$

$$\text{where } u=\frac{dx}{dt},\quad v=\frac{dy}{dt},\quad w=\frac{dz}{dt},$$

and we have

$$-(qw+v)\frac{d\frac{dx}{dt}}{dt}+(pw+u)\frac{d\frac{dy}{dt}}{dt}+(qu-qv)\frac{d\frac{dz}{dt}}{dt}$$

$$=R\left\{-(qw+v)\cos.\epsilon+(pw+u)\cos.\epsilon'+(qu-pv)\cos.\epsilon''\right\}.$$

On the right hand side put for cos. ϵ, cos. ϵ', cos. ϵ'', their values as already found (Art. 151.), and the expression will vanish. Also the left hand side is

$$p\left(w\frac{dv}{dt}-v\frac{dw}{dt}\right)+q\left(u\frac{dw}{dt}-w\frac{du}{dt}\right)+u\frac{dv}{dt}-v\frac{du}{dt}$$

$$=pw^2\frac{d\frac{v}{w}}{dt}+qu^2\frac{d\frac{w}{u}}{dt}+u^2\frac{d\frac{v}{u}}{dt}$$

$$=\left(pw^2\frac{d\frac{v}{w}}{dx}+qu^2\frac{d\frac{w}{u}}{dx}+u^2\frac{d\frac{v}{u}}{dx}\right)\frac{dx}{dt}.$$

Hence we have for the path described, dividing by $u^2\frac{dx}{dt}$,

$$p\frac{w^2}{u^2}\frac{d.\frac{v}{w}}{dx}+q\frac{d\frac{w}{u}}{dx}+\frac{d\frac{v}{u}}{dx}=0.$$

But $\frac{v}{u}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{dy}{dx}$; and similarly $\frac{w}{u}=\frac{dz}{dx}$; $\frac{v}{w}=\frac{dy}{dz}$.

Let $\dfrac{dy}{dx} = \phi$; also $\dfrac{dz}{dx} = p + q\dfrac{dy}{dx} = p + q\phi$.

Hence $\dfrac{v}{u} = \phi$, $\dfrac{w}{u} = p + q\phi$, $\dfrac{v}{w} = \dfrac{\phi}{p + q\phi}$.

And the equation becomes

$$p(p + q\phi)^2 \frac{d\dfrac{\phi}{p + q\phi}}{dx} + q\frac{d(p + q\phi)}{dx} + \frac{d\phi}{dx} = 0;$$

reducing, it becomes

$$p(p + q\phi)\frac{d\phi}{dx} - p\phi\frac{d(p + q\phi)}{dx} + q\frac{d(p + q\phi)}{dx} + \frac{d\phi}{dx} = 0;$$

$$(p^2 + pq\phi + 1)\frac{d\phi}{dx} - (p\phi - q)\left\{\frac{dp}{dx} + \phi\frac{dq}{dx} + q\frac{d\phi}{dx}\right\} = 0;$$

$$(1 + p^2 + q^2)\frac{d\phi}{dx} - (p\phi - q)\left(\frac{dp}{dx} + \phi\frac{dq}{dx}\right) = 0;$$

putting for p, q the functions of x, y which they denote,

$$\frac{d^2y}{dx^2} \text{ for } \frac{d\phi}{dx},\ \frac{dy}{dx} \text{ for } \phi,$$

this becomes a differential equation in x, y, which, along with the equations to the surface, determines the curve described.

The curve thus described is the shortest line which can be drawn from one point in it to another, upon the surface.

The velocity is constant, as appears from the equation

$$\text{velocity}^2 = 2\int_t \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right).$$

By methods somewhat similar we might determine the motion of a point upon a given curve of double curvature when acted upon by given forces.

CHAP. II.

ON THE CONSTRAINED MOTION OF SEVERAL POINTS.

Sect. I. GENERAL PRINCIPLES.

153. WHEN several points are connected so that they contain each other's motions, they exert forces upon one another; and so far as this is the case, the problem of their motions approaches to the problem of the motion of several free points mutually acting on each other, which has already been treated in the first Part of this Book. There is however generally this difference, that in that case the forces were given as explicit functions of the relative positions of the bodies; in this case the relative positions are given, and the forces are to be such as will secure these positions.

On the other hand, the problem of several connected points approaches that of a rigid body, which will be the subject of a succeeding Book. For two material points connected by a rigid rod may be considered as a body of finite magnitude. But in the problems of this Chapter, we shall consider only the cases where the mutual action of two points is in the straight line joining them, which limitation excludes the peculiar features of the problems of rigid bodies.

When two points act upon each other by means of a straight rod or string, connecting them, we may reduce the problems concerning their motion to the formulæ belonging to free motion, by considering the tension of the rod or string as one of the forces which act upon them. And as this tension will always be the same for both the bodies which this rod or cord connects, we shall be able to eliminate it, and thus to obtain the motions of the points.

The *tensions* here spoken of are of the nature of those which we have called *moving forces* or *pressures*. Consequently, the accelerating forces which they produce upon the bodies are, by the third law of motion, as the pressures or tensions directly, and as the mass of the bodies inversely. If the tension of a string which acts upon a body m be equal to the weight of a mass p, and if g be the *accelerating force* of gravity, we have

$$\frac{pg}{m} = \text{accelerating force produced by tension.}$$

We proceed to prove some general properties which obtain in the case of the constrained motions of a number of points.

Equality of Impressed and Effective Forces.

154. The forces which really act upon a system, are called the *impressed forces:* the forces which must act upon each of the points of the system, supposing them unconstrained, in order to produce the motions of those points which really take place, are called *the effective forces.*

PROP. *In all cases of the motion of points, connected so that the mutual action of any two is in the straight line which joins them*, the impressed and the effective forces are statically equivalent to each other.*

The motion of each point is produced by the impressed force acting on that point, and by the forces exerted by the other bodies upon that point, in virtue of the mutual action of the points. Let F be the impressed force on any point A, and R the force arising from the action of another point B

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* This condition includes all cases in which bodies are connected by flexible strings; for instance cases in which a string passes round a fixed point, and cases in which one string is fastened to another string by a knot or ring. The fixed point may be considered as a body of which the inertia is infinite; the knot or ring, as a body of which the inertia is nothing. But the condition excludes cases in which bodies are connected by rigid rods, and in which a rod passes through a fixed point, or in which two rods which meet are connected at a fixed angle. All such cases belong to Book III.

upon *A*. Let the impressed force *G* act upon *B*; in virtue of the mutual action of *A* and *B*, a reaction $-R$, equal and opposite to *R*, in the same straight line, will act upon *B*. In like manner let the impressed force *H* act on *C*, and let *S* be the action of *C* on *B*, and therefore $-S$ the action of *B* on *C*.

Now the effective force on *A* is that which arises from the composition of *F* and *R*; the effective force on *B* is that which arises from the composition of *G*, $-R$ and *S*; the effective force on *C* is that which arises from *H* and $-S$; and so on. And when all these effective forces are compounded, according to the rules for the statical composition of forces which act on any machine, *R* and $-R$ being equal and opposite will destroy each other, and also *S* and $-S$; and so of the other mutual forces. Therefore the whole resulting force will be that arising from the composition of *F*, *G*, *H*; that is, the resultant of the effective forces will be identical with the resultant of the impressed forces.

Conservation of Vis Viva.

155. The *vis viva* of any system of points, is the product of the mass of each point into the square of its velocity.

The *moving force* or *quantity of motion* of a body is generally understood to mean the product of the mass into the velocity, and is the same as the *momentum:* and the conservation of the quantity of force thus measured is proved by proving the conservation of the motion of the centre of gravity. But if the force of a body in motion be measured by the *whole* effect which it will produce before the velocity is destroyed, or by the whole effort which has been exercised in generating it, without regard to the time, it must be measured by the mass multiplied into the square of the velocity. Thus balls of the same size projected into a resisting substance, as a bed of clay, will go to the same depth so long as their weights, multiplied into the squares of their velocities, are the same. Force thus measured is called *vis viva*, in opposition to force measured by momentum, which is proportional to the pressure, or *dead pull*, producing it. And it will appear, that forces will always produce a certain quantity of *vis viva* by acting through a given space, whatever be the manner in which the bodies are constrained to move.

156. Prop. *The sum of the* vis viva *of each of the points of a system which has been acted upon by any forces, under the same condition as in* Art. 154., *is the same as if the points, being separate, had been acted upon by the same forces, through the same spaces.*

Let P be the impressed force which acts on the particle m, p the distance of m from a fixed point in the direction of this force; therefore $\frac{dp}{dt}$ is m's velocity in the direction of the force. Also, if q be the distance of m from a fixed point in the direction of its motion, $\frac{dq}{dt}$ is its velocity, $\frac{d^2q}{dt^2}$ the effective accelerating force, and $m\frac{d^2q}{dt^2}$ the effective moving force, by the principles of mechanics. And if we make similar suppositions with respect to m', m'', &c. we may consider the velocities of m, m', m'', &c. as virtual velocities, since they are consistent with the connexion of the system; and we have

$$\text{impressed forces, } Pm,\ P'm',\ P''m'', \text{ \&c.}$$

$$\text{with virtual velocities } \frac{dp}{dt},\ \frac{dp'}{dt},\ \frac{dp''}{dt}, \text{ \&c.}$$

$$\text{effective forces } m\frac{d^2q}{dt^2},\ m'\frac{d^2q'}{dt^2},\ m''\frac{d^2q''}{dt^2}, \text{ \&c.}$$

$$\text{with virtual velocities } \frac{dq}{dt},\ \frac{dq'}{dt},\ \frac{dq''}{dt}, \text{ \&c.}$$

Hence, since these forces must balance each other statically by last article, we have by the principle of virtual velocities, (see *Analytical Statics*, Art. 22.),

$$mP\frac{dp}{dt} + m'P'\frac{dp'}{dt} + \text{\&c.} = m\frac{dq}{dt}\frac{d^2q}{dt^2} + m'\frac{dq'}{dt}\frac{d^2q'}{dt^2} + \text{\&c.}$$

$$\text{or, } \Sigma . m \frac{dq}{dt} \frac{d^2q}{dt^2} = \Sigma . mP \frac{dp}{dt} \dots\dots\dots\dots (1).$$

And multiplying by 2 and integrating, and putting v for $\frac{dq}{dt}$,

$$\Sigma . mv^2 = C + 2\Sigma . m \int_p P \dots\dots\dots\dots\dots (2).$$

Now $2 \int_p P$ is the square of the velocity which the force P would have generated in a point separated from the rest; hence, the integral being taken between the same limits, the *vis viva* is the same as it would have been in that case.

157. Let x, y, z, be the co-ordinates of m, and we shall have the square of the velocity, or

$$v^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}.$$

Also, if X, Y, Z, be the resolved parts of P, parallel respectively to x, y, z, and if α, β, γ, be the co-ordinates of a fixed point O, in the direction in which this force acts, so that $Om = p$, we shall have

$$X = P . \frac{x - \alpha}{p}, \quad Y = P \frac{y - \beta}{p}, \quad Z = P . \frac{z - \gamma}{p}.$$

$$\text{And } p^2 = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2;$$

$$\therefore p = (x - \alpha) \frac{dx}{dp} + (y - \beta) \frac{dy}{dp} + (z - \gamma) \frac{dz}{dp}.$$

Hence,

$$X \frac{dx}{dp} + Y \frac{dy}{dp} + Z \frac{dz}{dp}$$

$$= \frac{P}{p} \left\{ (x - \alpha) \frac{dx}{dp} + (y - \beta) \frac{dy}{dp} + (z - \gamma) \frac{dz}{dp} \right\} = P.$$

By substituting these values, the equation before obtained becomes

$$\Sigma . m \left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\}$$

$$= C + 2\Sigma . m \left\{\int_p \left(X\frac{dx}{dp} + Y\frac{dy}{dp} + Z\frac{dz}{dp}\right)\right\}.$$

158. If the system be acted on by no forces, we shall have

$$\Sigma . mv^2 = C,$$

the sum of each particle multiplied into the square of its velocity will always be equal to a constant quantity.

If the system be acted on by gravity only, let x be vertical, and we shall have

$$\Sigma mv^2 = C - 2\Sigma . mgx = 2\Sigma . mg(h - x),$$

h being the height from which m must have fallen to acquire its velocity at any point.

This Proposition may be employed for the solution of a variety of Problems respecting bodies acted on by gravity.

The mutual action of the parts of a system may increase or diminish its *vis viva* in any degree. Their attraction or repulsion may augment the velocities, and consequently, the *vis viva*. Their collision will generally diminish the *vis viva*, except they be perfectly elastic, in which case, after the impact, the *vis viva* will be the same as before.

As the *vis viva* varies, it may become a *maximum* or *minimum*. This happens when the system passes through a position of equilibrium.

When the *vis viva* is a *maximum*, the body passes a position of *stable* equilibrium, when it is a *minimum*, it passes a position of *unstable* equilibrium See Poisson, *Traité de Mec.* No. 472, &c.

Principle of Least Action.

159. The *action* of a particle is here measured by the product of the momentum, and of the space through which the body moves. If the velocity be perpetually varying, the action will be the integral of this product taken for a differential of the space. If m be the mass, v the velocity, and s the space, the whole action of the body is $\int_s mv$, where the integral is to be taken between the beginning and end of the motion. And the principle to be proved is this:

PROP. *When a system moves in any manner, the actual motion is such, that the sum of the actions of each particle for the whole motion is less than if the particles had taken any other paths between the same points.*

That is, $\int_s \Sigma mv$ is a minimum.

To prove this, we must, according to the Calculus of Variations, prove that the *variation* of this integral, between the proper limits, is 0. See Lacroix, *Elem. Treat.* Art. 331.

In the preceding proof of the principle of *vis viva*, we have taken the actual velocities $\frac{dp}{dt}$, &c., $\frac{dq}{dt}$, &c. for the virtual velocities. But if δp, &c., δq, &c. be any possible corresponding small variations of p and q, they will be as the virtual velocities; and we shall have, instead of equation (1), p. 45, this equation,

$$\Sigma m \frac{d^2 q}{dt^2} \delta q = \Sigma m P \delta p \ldots\ldots\ldots\ldots (3).$$

Also, as in p. 45, it will appear, that

$$P = X\frac{\delta x}{\delta p} + Y\frac{\delta y}{\delta p} + Z\frac{\delta z}{dp},$$

$$\text{or,} \quad P\delta p = X\delta x + Y\delta y + Z\delta z.$$

And since $\frac{dq^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}$, differentiating, &c.

$$\frac{d^2q}{dt^2} \cdot \frac{dq}{dt} = \frac{d^2x}{dt^2}\,\frac{dx}{dt} + \frac{d^2y}{dt^2}\,\frac{dy}{dt} + \frac{d^2z}{dt^2}\,\frac{dz}{dt}.$$

Hence, considering the first side as the differential of a function of q, and the second as the differential of a function of x, y, z, we shall have the variations in the same manner, or

$$\frac{d^2q}{dt^2}\delta q = \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z.$$

Hence, equation (3) becomes

$$\Sigma . m \left\{\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2} dz\right\} = \Sigma . mP\delta p.$$

$$\text{Now } \delta\int_s \Sigma mv = \delta\int_t \Sigma m x v\frac{ds}{dt} = \int_t \Sigma m\delta . v\frac{ds}{dt};$$

$$\text{and } \Sigma . m\delta . v\frac{ds}{dt} = \Sigma . mv\delta\frac{ds}{dt} + \Sigma . m\frac{ds}{dt}\delta v.$$

$$\text{Now } \frac{ds}{dt} = v; \ \therefore \ \frac{ds}{dt}\delta v = v\delta v = \frac{1}{2} . \delta . v^2;$$

$$\therefore \ \Sigma . m\frac{ds}{dt}\delta v = \frac{1}{2}\Sigma . m\delta . v^2.$$

$$\text{But } \Sigma mv^2 = C + 2\Sigma . m\int_p P;$$

$$\therefore \ \Sigma . m\delta . v^2 = 2\Sigma . mP\delta p$$

$$= 2\Sigma . m\left\{\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right\}.$$

$$\text{Also } \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2};$$

$$\therefore \ \frac{ds}{dt}\delta\frac{ds}{dt} = \frac{dx}{dt}\delta\frac{dx}{dt} + \frac{dy}{dt}\delta\frac{dy}{dt} + \frac{dz}{dt}\delta\frac{dz}{dt},$$

$$\text{or } v\delta\frac{ds}{dt} = \frac{dx}{dt}\delta\frac{dx}{dt} + \frac{dy}{dt}\delta\frac{dy}{dt} + \frac{dz}{dt}\delta\frac{dz}{dt}.$$

Hence, since $\delta\dfrac{dx}{dt} = \dfrac{d\delta x}{dt}$ &c., adding the values of

$\Sigma m\dfrac{ds}{dt}dv$ and $\Sigma mv\delta\dfrac{ds}{dt}$, we have

$$\delta\Sigma mv\frac{ds}{dt} = \Sigma . m\left\{\begin{matrix}\dfrac{dx}{dt}\dfrac{d\delta x}{dt} + \dfrac{dy}{dt}\dfrac{d\delta y}{dt} + \dfrac{dz}{dt}\dfrac{d\delta z}{dt} \\ \dfrac{d^2x}{dt^2}\delta x + \dfrac{d^2y}{dt^2}\delta y + \dfrac{d^2z}{dt^2}\delta z\end{matrix}\right\}$$

$$= \Sigma . m\frac{d\left\{\dfrac{dx}{dt}\delta x + \dfrac{dy}{dt}\delta y + \dfrac{dz}{dt}\delta z\right\}}{dt}$$

$$= \frac{d . \Sigma . m\left\{\dfrac{dx}{dt}\delta x + \dfrac{dy}{dt}\delta y + \dfrac{dz}{dt}\delta z\right\}}{dt};$$

$$\therefore\ \delta\Sigma\int_s mv = \delta\int_t\Sigma mv\frac{ds}{dt} = \int_t\delta\Sigma mv\frac{ds}{dt}$$

$$= \Sigma . m\left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\}.$$

Now in this integration the value of the right-hand side is to be taken between the limits of the motion of the points m, m', &c.; and these limits are fixed, therefore δx, δy, δz, are each 0.

Hence, $\delta\int_t\Sigma mv\dfrac{ds}{dt} = 0$, and $\int_t\Sigma mv\dfrac{ds}{dt}$ is a *minimum*.

Cor. It follows from this, that the sum of the *vis viva* of a system in passing from one position to another, is a minimum. For $\int_t\Sigma mv\dfrac{ds}{dt} = \int_t\Sigma mv^2$. Hence, this sum of the *vis viva*, taken for each element of the time, for instance, for each second, is less than it would be if the system had moved in any other manner to the same position.

When the particles are not acted upon by any accelerating force, the *vis viva* is constant. Hence, the sum of the *vis viva* for any time is proportional to the time. And hence, in passing from one position to another, the time actually employed is less than it must have been if the body had followed any other path.

The principle of least action was announced by Maupertuis as a fundamental law of Mechanics, and was attempted to be proved *a priori*. It appears from what has been said, that it is a consequence of the elementary laws of motion.

160. In the remainder of this Chapter it is intended to consider some problems in which several material points are connected with each other by means of cords or rods, not possessing inertia or weight. These cords or rods will in general be supposed inflexible and inextensible: also, we may suppose that one or more of the points are compelled to move in given lines or surfaces.

Sect. II. THE MOTION OF A ROD ON PLANES*.

PROB. I. Fig. 9. P, Q, *are two material points, connected by an inflexible rod* PQ; PQ *falls by gravity in a vertical plane, while* P *moves along a horizontal plane: to define the motion of* Q.

N.B. If PQ have at first no motion, except in a vertical plane, it will continue to move in a vertical plane, because all the forces are in the same plane.

Let APN be the horizontal line, G the centre of gravity of P, Q; GH, QN vertical; let A be a fixed point, $AP = x$, $AN = x'$, $NQ = y'$. And let p = the tension of the rod PQ; (that is, the force with which it presses P in the direction QP, and Q in the direction PQ;) also, let θ be the angle NPQ; then the resolved parts of the tension will be $p \, . \cos . \theta$ parallel to AN, and $p\,.\sin.\theta$ parallel to NQ. And, if m, m', be the masses of P, Q, respectively, the accelerating forces will be

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* The motion of a rod without weight connecting two heavy bodies, will be the same as the motion of a heavy rod: this will appear hereafter.

$$-\frac{pg\cos.\theta}{m} \text{ on } P; \ \frac{pg\cos.\theta}{m'}, \text{ and } \frac{pg\sin.\theta}{m'} \text{ on } Q.$$

Hence, we have, by equations (a), Art. 19.

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} = -\frac{pg\cos.\theta}{m}, \\ &\frac{d^2x'}{dt^2} = \frac{pg\cos.\theta}{m'}, \ \frac{d^2y'}{dt^2} = \frac{pg\sin.\theta}{m'} - g \end{aligned}\right\} \ldots\ldots (1);$$

$$\therefore\ m\frac{d^2x}{dt} + m'\frac{d^2x'}{dt^2} = 0.$$

$$\text{Let } \bar{x} = AH = \frac{mx + m'x'}{m + m'};\ \therefore\ \frac{d^2\bar{x}}{dt^2} = 0;$$

hence, the point H will either remain at rest, or move uniformly along AN.

If we make $PG = b$, $QG = c$,

$$\bar{x} = AP = AH - PH = \bar{x} - b\cos.\theta;\ \therefore\ \frac{d^2\bar{x}}{dt^2} = -b\frac{d^2\cos.\theta}{dt^2},$$

$$x' = AN = AH + HN = \bar{x} + c\cos.\theta;\ \therefore\ \frac{d^2x'}{dt^2} = c\frac{d^2\cos.\theta}{dt^2}.$$

$$\text{Also, } y' = QN = \frac{QP}{PG}\,.\,GH = \frac{m + m'}{m'}\,b\sin.\theta.$$

Hence, the first and last of equations (1) give, substituting these values, and multiplying by m, m',

$$mb\frac{d^2\cos.\theta}{dt^2} = pg\cos.\theta;$$

$$(m + m')\,b\,.\frac{d^2\sin.\theta}{dt^2} = pg\sin.\theta - gm'.$$

Multiply the first of these equations by

$$2\frac{d.\cos.\theta}{dt} = -2\sin.\theta\frac{d\theta}{dt},$$

and the second by

$$2\frac{d.\sin.\theta}{dt} = 2\cos.\theta\frac{d\theta}{dt},$$

and add them, and we have

$$2mb\frac{d\cos.\theta}{dt}.\frac{d^2\cos.\theta}{dt^2} + 2(m+m')b\frac{d\sin.\theta}{dt}.\frac{d^2\sin.\theta}{dt^2}$$
$$= -2gm'\cos.\theta\frac{d\theta}{dt}.$$

This integrated, gives

$$mb\left(\frac{d.\cos.\theta}{dt}\right)^2 + (m+m')b\left(\frac{d\sin.\theta}{dt}\right)^2 = C - 2gm'\sin.\theta:$$

$$\text{or, } \{m\sin.^2\theta + (m+m')\cos.^2\theta\}b.\frac{d\theta^2}{dt^2} = C - 2gm'\sin.\theta:$$

which reduced, gives

$$b.\frac{d\theta^2}{dt^2} = \frac{C-2gm'\sin.\theta}{m+m'\cos^2.\theta}.$$

To determine C, suppose that when PQ reaches the line AN, or when $\theta = 0$, the angular velocity $\frac{d\theta}{dt}$ is a: hence,

$$ba^2 = \frac{C}{m+m'};$$

$$\therefore \frac{d\theta^2}{dt^2} = \frac{ba^2(m+m') - 2gm'\sin.\theta}{b(m+m'\cos.^2\theta)} \ldots\ldots\ldots(2).$$

There may be a position when the angular velocity is 0; suppose that there θ is θ_1; then, we shall have

$$ba^2(m+m') - 2gm\sin.\theta_1 = 0 \ldots\ldots\ldots(3);$$

$$\frac{d\theta^2}{dt^2} = \frac{2gm}{b}.\frac{\sin.\theta_1 - \sin.\theta}{m+m'\cos.^2\theta}.$$

If the rod fall from being at rest in a given position, θ_1 is known: and hence the angular velocity at any point. To determine the position at any time, we must obtain t in terms of θ, by integrating again; and hence θ in terms of t.

If the line fall from a vertical position $\theta_1 = \frac{\pi}{2}$; hence, to find the angular velocity a, acquired when it becomes horizontal, we have, by (3),

$$a^2 = \frac{2gm}{b(m+m')}; \quad \therefore \; b^2 a^2 = \frac{2bgm}{m+m'}.$$

Now ba is the linear velocity of the point G at H, and, it appears from this, that the space through which a body must fall freely to acquire this velocity, is $\frac{bm'}{m+m'}$, a third proportional to QP and GP.

If it were supposed that P were constrained to move in a horizontal groove AP, while PQ were so connected with it, that PQ could descend below the horizontal position, so as to revolve entirely round P; the mechanical conditions would be the same, and the expression obtained above for the angular velocity is true in this case also.

$$\text{Since } \frac{d\theta}{dt} = \sqrt{\frac{ba^2(m+m') - 2m'g\sin.\theta}{b(m+m'\cos.^2\theta)}},$$

the motion is possible, so long as the numerator of the fraction is positive. The negative part is greatest, when $\theta = \frac{\pi}{2}$; hence, the numerator is always positive, if

$$ba^2(m+m') > 2mg, \text{ or if } a^2 > \frac{2mg}{(m+m')b}.$$

In this case, if H be at rest at first, the line PQ will go on revolving about the point H; the centre of gravity G ascending and descending in a vertical line.

If H be at rest, the path described by Q will be an ellipse, whose semi-axes are GQ and PQ.

$$\text{For } \left(\frac{GO}{GQ}\right)^2 + \left(\frac{QN}{PQ}\right)^2 = 1, \text{ or } \frac{x^2}{c^2} + \frac{y^2}{(b+c)^2} = 1:$$

the equation to an ellipse, of which the semi-axes are $b+c$ and c.

$$\text{The velocity of } P = b\frac{d.\cos.\theta}{dt} = -b\sin.\theta.\frac{d\theta}{dt}$$

$$= -b\sin.\theta\sqrt{\frac{ba^2(m+m') - 2m'g\sin.\theta}{b(m+m'\cos^2.\theta)}} \ldots\ldots (4).$$

If H be in motion, and move uniformly so as to carry the whole line PQ in the direction AX, the relative motion of the points P, Q will not be altered. The system will have the motion of rotation already investigated, combined with the uniform motion of translation in the direction AX; and the point Q by the mixture of its elliptical motion with this rectilinear one, will describe a kind of trochoidal curve.

Cor. 1. *Retaining the conditions of the problem, suppose* Q *to hang vertically from* P, *and a given velocity* V *to be communicated to* P: *to determine the motion of* P *and* Q.

The velocity of G parallel to AX is constant, because $\frac{d^2\bar{x}}{dt^2} = 0$, and $\frac{d\bar{x}}{dt} =$ constant. Hence, G will retain the velocity parallel to AX, which it has at first. But at first, when Q is at rest, and P moves with the velocity V, the velocity of G parallel to AX will be $\frac{V.QG}{QP} = \frac{Vc}{b+c}$. And at first the angular velocity of Q round P will be $\frac{V}{b+c}$; hence, by (2), putting $\frac{\pi}{2}$ for θ,

$$\frac{V^2}{(b+c)^2} = a^2.\frac{m+m'}{m} - \frac{2gm'}{mb};$$

whence, a is known; and hence the motion of the system is completely determined.

The quantity g, which represents the vertical force may be of any value. If we make it $=0$, we have the motion of a rod moving freely (as it would do on a smooth horizontal plane,) while one end moves in a rectilinear groove. In this case

$$\frac{d\theta^2}{dt^2}=\frac{a^2(m+m')}{m+m'\cos.^2\theta}.$$

This velocity is a, when the rod coincides with the groove, and increases to $a\sqrt{\frac{m+m'}{m}}$ when it is perpendicular to it.

If we make $m=0$, we have, by (2)

$$\frac{d\theta^2}{dt^2}=\frac{ba^2-2g\sin.^2\theta}{b\cos.\theta}.$$

Here H coincides with N; we might find θ by integrating. It is easily seen that the curve described is a parabola.

161. We have deduced the solution of this problem directly from the equations of motion. In many such cases however the principle of the conservation of *vis viva* gives us at once one integral of these equations.

This may be exemplified in the problem now under consideration.

Retaining the previous notation, we have, for the (velocity)^2 of the point Q,

$$\frac{dx'^2}{dt^2}+\frac{dy'^2}{dt^2}=\left\{c^2\sin.^2\theta+\left(\frac{m+m'}{m'}\right)^2b^2\cos.^2\theta\right\}\frac{d\theta^2}{dt^2}.$$

Also, for the (velocity)^2 of P, we have $b^2\sin^2.\theta\frac{d\theta^2}{dt^2}$.

The only force which has produced motion, has been gravity acting on Q: and if h be the original height, $h - \frac{h - m'}{m} b \sin. \theta$ is the space fallen through by Q.

Hence, the principle of the conservation of *vis viva* gives, observing that $m'c = mb$

$$\left\{\frac{b^2}{m'} m^2 \sin.^2 \theta + (m + m')^2 \cos.^2 \theta + m b^2 \sin^2. \theta\right\} \frac{d\theta^2}{dt^2}$$

$$= 2 m' g \left\{h - \frac{m + m'}{m'} b \sin^2. \theta\right\}:$$

whence

$$b \left\{m \sin^2 \theta + (m + m') \cos.^2 \theta\right\} \frac{d\theta^2}{dt^2} = \frac{2 m'^2 g h}{m + m'} - 2 m' g \sin. \theta,$$

which agrees with the equation obtained by integration before.

Prob. II. *The same things being supposed, except that the plane* AX *is inclined to the horizon; to determine the motion.*

The rod is still supposed to move in a vertical plane.

If we resolve the forces parallel and perpendicular to the plane AX, we shall find that the point H will descend down the inclined plane in the same manner as a heavy point would do; that is, acted on by a constant force $g \sin. \iota$, ι being the inclination: and that the angular motion will be obtained in the same manner as in the last problem, except that instead of g we shall have $g \cos. \iota$, the force perpendicular to the plane.

If the line AX be vertical, the motion of PQ, with respect to AX, will be the same as it was in the last problem when we made $g = 0$.

Prob. III. *A rod* PQ, *connecting two material points* P, Q, *acted on by gravity, moves so that one of them slides along a horizontal plane: to define the motion.*

This differs from Prob. I, in not supposing the motion to be in a vertical plane.

Let AL, LP, fig. 10, be rectangular co-ordinates of P in the horizontal plane, AM, MN, NQ co-ordinates of Q,

$$AL = x, \quad LP = y, \quad AM = x', \quad MP = y', \quad NQ = z', \quad PQ = b.$$

Also let $NOM = \phi$, and $QPN = \theta$; hence,

$$z' = b \sin.\theta, \quad x' - x = b \cos.\theta \cos.\phi, \quad y' - y = b \cos.\theta \sin.\phi.$$

Let p be the tension of PQ, m, m' the masses of P, Q.

And resolving the forces, we have

$$\left.\begin{array}{l} \dfrac{d^2x}{dt^2} = -\dfrac{pg}{m} \cos.\theta \cos.\phi, \quad \dfrac{d^2y}{dt^2} = -\dfrac{pg}{m} \cos.\theta \sin.\phi, \\ \dfrac{d^2x'}{dt^2} = \dfrac{pg}{m} \cos.\theta \cos.\phi, \quad \dfrac{d^2y'}{dt^2} = \dfrac{pg}{m'} \cos.\theta \sin.\phi, \\ \qquad\qquad \dfrac{d^2z'}{dt^2} = \dfrac{pg}{m'} \sin.\theta - g. \end{array}\right\} \ldots\ldots(1).$$

Hence, we have $m \dfrac{d^2x}{dt^2} + m' \dfrac{d^2x'}{dt^2} = 0$; $m \dfrac{d^2y}{dt^2} + m' \dfrac{d^2y'}{dt^2} = 0$.

From which it appears that the projection of the centre of gravity of P, Q, on the plane of xy, moves in a straight line with a uniform velocity.

$$\left.\begin{array}{l} \text{Also } \dfrac{d^2x'}{dt^2} - \dfrac{d^2x}{dt^2} = \left(\dfrac{1}{m} + \dfrac{1}{m'}\right) pg \cos.\theta \cos.\phi, \\ \qquad \dfrac{d^2y'}{dt^2} - \dfrac{d^2y}{dt^2} = \left(\dfrac{1}{m} + \dfrac{1}{m'}\right) pg \cos.\theta \sin.\phi \end{array}\right\} \ldots(2).$$

And $y' - y = b \cos.\theta \sin.\phi$, $x' - x = b \cos.\theta \cos.\phi$;

$$\therefore \quad (x' - x) \frac{d^2(y' - y)}{dt^2} - (y' - y) \frac{d^2(x' - x)}{dt^2} = 0:$$

whence it appears that N describes about P areas proportional to the times; (see Art. 22,) and therefore

$$b^2 \cos.^2\theta \frac{d\phi}{dt} = h: \ h \text{ being a constant quantity} \ldots\ldots(3).$$

Again, equations (2) may be put in this form

$$\frac{d^2 . \cos . \theta \cos . \phi}{dt^2} = \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{pg}{b} \cos . \theta \cos . \phi ;$$

$$\frac{d^2 . \cos . \theta \sin . \phi}{dt^2} = \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{pg}{b} \cos . \theta \sin . \phi .$$

And multiplying by cos. ϕ, sin. ϕ, and adding

$$\cos . \phi \frac{d^2 . \cos \theta \cos . \phi}{dt^2} + \sin . \phi \frac{d^2 . \cos . \theta \sin . \phi}{dt^2}$$

$$= \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{pg}{b} \cos . \theta .$$

But $\frac{d . \cos . \theta \cos . \phi}{dt} = \cos . \phi \frac{d . \cos . \theta}{dt} - \cos \theta \sin . \phi \frac{d\phi}{dt}$,

$$\frac{d^2 . \cos . \theta \cos . \phi}{dt^2} = \cos . \phi \frac{d^2 \cos . \theta}{dt^2} - 2 \sin . \phi \frac{d\phi}{dt} \frac{d \cos . \theta}{dt}$$

$$- \cos . \theta \cos . \phi \frac{d\phi^2}{dt^2} - \cos . \theta \sin . \phi \frac{d^2\phi}{dt^2},$$

$$\frac{d . \cos . \theta \sin . \phi}{dt} = \sin . \phi \frac{d \cos . \theta}{dt} + \cos . \theta \cos . \phi \frac{d\phi}{dt},$$

$$\frac{d^2 . \cos . \theta \sin . \phi}{dt^2} = \sin . \phi \frac{d^2 \cos . \theta}{dt^2} + 2 \cos . \phi \frac{d\phi}{dt} \frac{d \cos . \theta}{dt}$$

$$- \cos . \theta \sin . \phi \frac{d\phi^2}{dt^2} + \cos . \theta \cos . \phi \frac{d^2\phi}{dt^2} .$$

Hence, $\cos . \phi \frac{d^2 . \cos . \theta \cos . \phi}{dt^2} + \sin . \phi \frac{d^2 \cos . \theta \sin . \phi}{dt^2}$

becomes $\frac{d^2 \cos . \theta}{dt^2} - \cos . \theta \frac{d\phi^2}{dt^2}$;

or since, by (3), $\frac{d\phi^2}{dt^2} = \frac{h^2}{b^4 \cos .^4 \theta}$, it becomes

$$\frac{d^2 \cos . \theta}{dt^2} - \frac{h^2}{b^4 \cos .^3 \theta}$$

And the above equation is equivalent to

$$\frac{d^2 \cos.\theta}{dt^2} - \frac{h^2}{b^4 \cos.^3\theta} = \left(\frac{1}{m} + \frac{1}{m'}\right)\frac{pg}{b}\cos.\theta \ldots\ldots (4).$$

Also the equation in x' may be written thus,

$$\frac{d^2 \sin.\theta}{dt^2} = \frac{pg}{bm'} . \sin.\theta - \frac{g}{b} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Eliminate p by multiplying these by $m \sin.\theta$, and $(m+m') \cos.\theta$, and subtracting, hence

$$(m+m')\cos.\theta \frac{d^2 \sin.\theta}{dt^2} - m \sin.\theta \frac{d^2 \cos.\theta}{dt^2} - \frac{mh^2 \sin.\theta}{b^4 \cos.^3\theta}$$

$$= -(m+m')\frac{g}{b}\cos.\theta.$$

Multiply by $2\dfrac{d\theta}{dt}$, and we may put the equation in this form,

$$2(m+m')\frac{d \sin.\theta}{dt}\frac{d^2 \sin\theta}{dt^2} + 2m \frac{d \cos.\theta}{dt}\frac{d^2 \cos.\theta}{dt^2}$$

$$+ \frac{2mh^2}{b^4 \cos.^3\theta}\frac{d \cos.\theta}{dt} = -\frac{2(m+m')g}{b}\frac{d . \sin.\theta}{dt}.$$

Integrating, $(m+m')\left(\dfrac{d \sin.\theta}{dt}\right)^2 + m\left(\dfrac{d \cos.\theta}{dt}\right)^2 - \dfrac{mh^2}{b^4 \cos.^2\theta}$

$$= C - \frac{2(m+m')g}{b}\sin.\theta;$$

$$\therefore \frac{d\theta^2}{dt^2}\{(m+m')\cos.^2\theta + m \sin.^2\theta\}$$

$$= \left\{C - \frac{2(m+m')g \sin.\theta}{c} + \frac{mh^2}{b^4 \cos.^2\theta}\right\}$$

$$\frac{d\theta^2}{dt^2}\{m+m' \cos.^2\theta\} = C - \frac{2(m+m') . g \sin.\theta}{b} + \frac{mh^2}{b^4 \cos.^2\theta} \ldots\ldots (6).$$

When $\theta = 0$, $\frac{d\theta^2}{dt^2}(m + m') = m\,C + \frac{mh^2}{b^4}$.

We can never have $\theta = \frac{\pi}{2}$; for then $\frac{d\theta}{dt}$ is infinite.

To find when $\frac{d\theta}{dt} = 0$, we have a cubic equation: but it is manifest that if θ be a value which satisfies this equation, $\pi - \theta$ will also satisfy it.

Having found θ in terms of t by (6), we should find ϕ by equation (3). From equation (3) $\frac{d\phi}{d\theta}\frac{d\theta}{dt} = \frac{h}{b^2 \cos.^2 \theta}$; and if we eliminate $\frac{d\theta}{dt}$ by (6), we have $\frac{d\phi}{d\theta} =$ a function of θ. And if we put

$$b \cos.\theta = PN = r, \quad \frac{d\theta}{dr} = -\frac{1}{\sqrt{(b^2 - r^2)}},$$

whence $\frac{d\phi}{dr} =$ a function of r, which is the equation to the orbit of N about P.

If $m = 0$, we have

$$\frac{d\theta^2}{dt^2} \cdot m' \cdot \cos.^2 \theta = C - \frac{2m'g}{b} \sin.\theta.$$

And when $\theta = 0$, $\frac{d\theta^2}{dt^2} = \frac{C}{m'}$.

In this case we might integrate. The body Q will describe a parabola.

Prob. IV. *Two points* P, Q, *fig.* 11, *are connected by a rod, and one of them* P, *slides along a vertical line* AZ: *to define the motion.*

Let $AP = z$; and let $AM = x'$, $MN = y'$, $NQ = z'$, three rectangular co-ordinates of the point Q from a fixed point A.

Let $QPZ = \theta$, $MAN = \phi$, $PQ = b$;

$\therefore$ $x' = b \sin.\theta . \cos.\phi$, $y = b \sin.\theta . \sin.\phi$, $z' - z = b \cos.\theta$.

And if p be the tension of PQ, the equations of motion will be

$$\left.\begin{array}{l} \dfrac{d^2x'}{dt^2} = \dfrac{pg}{m'} \sin.\theta \cos.\phi, \quad \dfrac{d^2y'}{dt^2} = \dfrac{pg}{m'} \sin.\theta \sin.\phi \\ \dfrac{d^2z'}{dt^2} = g + \dfrac{pg}{m'} \cos.\theta, \quad \dfrac{d^2z}{dt^2} = g - \dfrac{pg}{m} \cos.\theta \end{array}\right\} \ldots\ldots(1).$$

Hence, as in last problem, we find

$$x'\frac{d^2y'}{dt^2} - y'\frac{d^2x'}{dt^2} = 0, \text{ and } b^2 \sin.^2\theta \frac{d\phi}{dt} = h \ldots\ldots\ldots(2).$$

Also, $\cos.\phi \dfrac{d^2x'}{dt^2} + \sin.\phi \dfrac{d^2y'}{dt^2} = \dfrac{pg}{m'} \sin.\theta$,

which, transformed as in last problem, gives

$$\frac{d^2 \sin.\theta}{dt^2} - \sin.\theta \frac{d\phi^2}{dt^2} = \frac{pg}{m'b} . \sin.\theta;$$

$$\therefore \text{ by } (2), \quad \frac{d^2 \sin.\theta}{dt^2} - \frac{h^2}{b^4 \sin.^3\theta} = \frac{pg}{m'b} \sin.\theta \ldots\ldots\ldots(3).$$

And $b \dfrac{d^2 \cos.\theta}{dt^2} = \dfrac{d^2z'}{dt^2} - \dfrac{d^2z}{dt^2}$; hence, by equations (1),

$$\frac{d^2 \cos.\theta}{dt^2} = \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{pg}{b} \cos.\theta \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Eliminating p, as in last problem, we obtain

$$2(m + m') \frac{d \sin.\theta}{dt} \frac{d^2 \sin.\theta}{dt^2} + 2m \frac{d \cos.\theta}{dt} \frac{d^2 \cos\theta}{dt^2}$$

$$- \frac{2h^2(m + m')}{b^4 \sin.^3\theta} \frac{d \sin.\theta}{dt} = 0.$$

Integrating, $$(m+m')\left(\frac{d\sin.\theta}{dt}\right)^2+\left(\frac{d\cos.\theta}{dt}\right)^2$$
$$+\frac{h^2(m+m')}{b^4\sin.^2\theta}=C,$$

$$\frac{d\theta^2}{dt^2}.(m+m'\cos.^2\theta)=C-\frac{h^2(m+m')}{b^4\sin.^2\theta}\dots\dots\dots\dots(5).$$

Hence, the angular velocity of PQ in a vertical direction is known. And hence, by (2), we may, as in last problem, find the differential equation to the orbit described by N round A. AN describes areas proportional to the times by (2).

The relation between t and θ may be expressed by means of an elliptical arc. Equation (5) may be put in this form,

$$\frac{dt^2}{d\theta^2}=\sin.^2\theta.\frac{m+m'\cos.^2\theta}{C-\frac{h^2(m+m')}{b^4}-C\cos.\theta^2}$$

$$=\sin.^2\theta.\frac{m+m'\cos.^2\theta}{D-C\cos.^2\theta}.$$

Making $D=C-\frac{h^2(m+m')}{b^4}$.

Now, if 1 and a be the semi-axis minor and major of an ellipse, σ the arc measured from the extremity of the semi-axis, and ξ its abscissa measured from the centre along the axis-minor; we shall have

$$\frac{d\sigma^2}{d\xi^2}=\frac{1+(a^2-1)\xi^2}{1-\xi^2}.$$

If, in the value of $\frac{dt^2}{d\theta^2}$ we make $C\cos.^2\theta=D\xi^2$, we have

$$\cos.\theta=\sqrt{\frac{D}{C}}.\xi;\quad -\sin.\theta\frac{d\theta}{d\xi}=\sqrt{\frac{D}{C}};$$ and hence,

$$\frac{dt^2}{d\xi^2} = \frac{dt^2}{d\theta^2}\frac{d\theta^2}{d\xi^2} = \frac{D}{C}\cdot\frac{m + \dfrac{Dm'}{C}\xi^2}{D - D\xi^2} = \frac{m}{C}\cdot\frac{1 + \dfrac{Dm'}{Cm}\xi^2}{1 - \xi^2};$$

which agrees with $\frac{m}{C}\frac{d\sigma^2}{d\xi^2}$, if we make

$$a^2 - 1 = \frac{Dm'}{Cm},$$ whence a is known. Hence

$$t + C' = \sqrt{\frac{m}{C}} \times \text{elliptical arc} \left(\text{abscissa} = \cos.\theta\sqrt{\frac{C}{D}}\right);$$

$$\therefore\ \cos.\theta\sqrt{\frac{C}{D}} = \text{abscissa of elliptical arc } (t + C)\sqrt{\frac{C}{m}}.$$

When $\theta = \frac{\pi}{2}$, $\xi = 0$, and the arc $= 0$, and if at that time t is t_1, $C = -t_1$.

Also, when $\theta = \frac{3\pi}{2}$, $\xi = 0$, and the arc = a semi-ellipse; hence, $\tau\sqrt{\frac{C}{m}}$ = semi-ellipse, gives the interval τ between two horizontal positions of the rod.

When $\theta = 0$, the angular velocity is infinite; hence, the rod will never coincide with the vertical line.

This is true only if the body have some angular motion horizontally, for if it move at first in a vertical plane, it will continue to do so, and its motion may be calculated by Prob. III.

PROB. V. *Two points* P, Q, *connected by a rod, slide along two given inclined planes, acted on by gravity; to determine the motion.*

The motion is supposed to take place in a vertical plane.

Let AX, AY, fig. 12, be the two planes, making with the horizon angles β, γ. Let PQ make an angle θ with AX, and

η with AY. $AP = x$, $AQ = y$, $PQ = b$; tension of $PQ = p$, masses of P, $Q = m$, m'.

Resolving the forces in the lines AX, AY, we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{pg}{m}\cos.\theta - g\sin.\beta \\ \frac{d^2y}{dt^2} &= \frac{pg}{m'}\cos.\eta - g\sin.\gamma \end{aligned}\right\}\text{......}(1).$$

$$\text{Hence, } m'\cos.\eta\frac{d^2x}{dt^2} - m'\cos.\theta\frac{d^2y}{dt^2}$$

$$= g\,(m'\sin.\gamma\cos.\theta - m\sin.\beta\cos.\eta)\text{......}(2).$$

$$\text{Now } x = \frac{b\sin.\eta}{\sin.(\beta+\gamma)},\quad y = \frac{b\sin.\theta}{\sin.(\beta+\gamma)}.$$

Substituting these values in (2), multiplying some of the terms by 2, and others by $-2\frac{d\eta}{d\theta}$, (since $\frac{d\eta}{d\theta} = -1$) and integrating, we have

$$\frac{b}{\sin.(\beta+\gamma)}\left\{m\left(\frac{d.\sin.\eta}{dt}\right)^2 + m'\left(\frac{d.\sin.\theta}{dt}\right)^2\right\}$$

$$= C - 2g\,(m'\sin.\gamma\sin.\theta + m\sin.\beta\sin.\eta);$$

$$\therefore\ \frac{b}{\sin.(\beta+\gamma)}\cdot\frac{d\theta^2}{dt^2}\cdot(m\cos.^2\eta + m'\cos.^2\theta)$$

$$= C - 2g\,(m'\sin.\gamma\sin.\theta + m\sin.\beta\sin.\eta)\text{......}(3).$$

Hence, the angular velocity is known.

This may be reduced as follows. Let the angle which PQ makes with the horizon be ψ. Then, $\theta = \beta + \psi$, $\eta = \gamma - \psi$. Also, let $C = 2gn$; then the right-hand side of equation (3) becomes

$$2g\,\{n - m'\sin.\gamma.\sin.(\beta+\psi) - m\sin.\beta.\sin.(\gamma-\psi)\}:$$

expanding, the part in brackets becomes

$$n - (m' + m) \sin. \beta \sin. \gamma \cos. \psi$$
$$- (m' \sin. \gamma \cos. \beta + m \sin. \beta \cos. \gamma) \sin. \psi.$$

$$\left.\begin{array}{r}\text{Suppose } (m + m') \sin. \beta \sin. \gamma = r \cos. \delta \\ m' \sin. \gamma \cos. \beta - m \sin. \beta \cos. \gamma = r \sin. \delta\end{array}\right\} \ldots\ldots\ldots (4),$$

which always give possible values of r and δ; and our expression becomes

$$n - r \cos. \delta \cos. \psi - r \sin. \delta \sin. \psi, \text{ or } n - r \cos. (\psi - \delta).$$

Hence, equation, (3) gives

$$\frac{d\theta^2}{dt^2} = \frac{2g \sin. (\beta + \gamma)}{b} \cdot \frac{n - r \cos. (\psi - \delta)}{m \cos.^2 (\gamma - \psi) + m' \cos.^2 (\beta + \psi)}.$$

By equations (4), $\tan. \delta = \dfrac{m' \cot. \beta - m \cot. \gamma}{m + m'}$;

and hence, by Mechanics, (Chap. V, Prob. 13.) δ is the value of ψ in the position of equilibrium of PQ. When $\psi = \delta$, the angular velocity is greatest. About this position PQ oscillates both ways; (except its velocity be too great.) We shall find the limit of the oscillations by making

$$\frac{d\theta}{dt} = 0; \ \therefore n - r \cos. (\psi - \delta) = 0; \ \cos. (\psi - \delta) = \frac{n}{r}.$$

And $\dfrac{n}{r}$ is the cosine of a positive arc, and of a negative arc of equal magnitude. Hence, the limiting positions PQ, $P'Q'$, make equal angles with pg the position of equilibrium.

If we know the velocity at any point, we know C, and consequently n, by equation (3); r is known by equation (4).

If the velocity be so great that n exceeds r, there will no longer be a limit to the angular motion of the rod. If we suppose XA, YA to be grooves in which P, Q, slide, instead of planes; and to be produced beyond *A*, which will not alter the mathematical conditions; the rod *PQ* will perform complete revolutions, and the limits to its positions will be curves *XYX'Y'*, fig. 13. The end *P* will oscillate backwards and forwards through *XX'*, and *Q* through *YY'*.

If we suppose *P*, *Q* to move in two grooves, not affected by gravity, we must in equation (3), make $g = 0$. Hence,

$$\frac{b}{\sin.(\beta+\gamma)} \cdot \frac{d\theta^2}{dt^2}(m\cos.^2\eta + m'\cos.^2\theta) = C.$$

If the grooves cross each other at right angles, and $m = m'$, *PQ* will revolve with a uniform angular velocity.

Prob. VI. *The rod* PQ *descends, one end sliding along a horizontal and another along a vertical plane: to determine its motion.*

The rod is supposed to move in a vertical plane.

This is a particular case of last problem. It might be simplified by beginning with the equations (1) properly modified; but it will be sufficient to apply our equation (3). Making, in that, $\beta = 0$, $\gamma =$ right angle, $\eta =$ comp. θ; we have

$$b \cdot \frac{d\theta^2}{dt^2}(m\sin.^2\theta + m'\cos.^2\theta) = C - 2gm'\sin.\theta.$$

If we suppose a to be the angular velocity when the rod is horizontal, or when $\theta = 0$;

$$ba^2m' = C.$$

If we suppose that when the angular velocity is 0, θ is θ_1,

$$0 = C - 2gm'\sin.\theta_1; \quad \therefore\ ba^2 = 2g\sin.\theta_1.$$

Hence, the angular velocity a acquired by falling from rest through an angle θ, to a horizontal position, is independent of m, m'.

When Q comes to the horizontal plane

$$\text{velocity}^2 \text{ of } Q = b^2 a^2 = 2gb \sin.\theta_1;$$

hence, the velocity is the same as if Q had fallen freely through the space $b \sin.\theta_1$, which is its actual descent.

If $ba^2 > 2g$, there is no position where the velocity $= 0$, and the rod will *revolve* in the manner described in the last problem.

Sect. III. TRACTORIES.

162. When one end of a string or rod is drawn with a given velocity along a given straight line or curve, (which may be called the *Directrix*,) so that a body fastened to the other end is dragged along by it, and made to describe a certain path, this path is called a *Tractory*. The motion of the body at any instant will depend upon its preceding motion, and upon the tension of the string. The body is considered as a material point.

PROB. VII. *A point* P, *fig.* 14, *moves uniformly along a straight line* AX, *drawing along with it, in the same plane, a body* Q, *by means of a string* PQ: *to find the tractory described by* Q *.

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* This problem is sometimes solved without taking account of the tendency which the angular motion, generated in PQ, has to continue. This mode of considering it would be true, if we were to consider Q to move upon a plane, and the friction to be such, as instantly to destroy any motion communicated to the body. In that case, Q, fig. 15, would always move in the direction QP, in which the string draws it; QP would be a tangent; and the curve would be determined by the condition, that the tangent QP, intercepted by the abscissa, is a constant quantity. Hence, if $AN = x$, $NQ = y$, rectangular co-ordinates; $QP = a$; we have

$$NP = -y\frac{dx}{dy}; \quad \therefore \; QP = a = y\sqrt{\left(1 + \frac{dx^2}{dy^2}\right)};$$

$$\therefore \; \frac{dx}{dy} = \frac{\sqrt{(a^2 - y^2)}}{y};$$

and by integrating we get the relation of x and y. (See Mr Peacock's Examples, p. 174, for the properties of this curve.)

This curve has an asymptote, as represented in fig. 15. We may reserve for it the name of *Tractrix*, by which it is frequently designated; giving to the curves, which really

We might solve this by the equations of motion: but we may find the curve more simply by the following reasoning.

If, when any system is in motion, we suppose the space in which it is, to move uniformly, so as to carry all the parts of the system in parallel directions, and with equal velocities; the relative motion of the parts, and their action upon one another will remain the same as before, by the second law of motion.

While *P* is moving with a uniform velocity along the line *AX*, let this line and the space containing *PQ* move in the opposite direction *XA*, with an equal uniform velocity. Therefore, by what has been said, the angular motion of *PQ* will remain the same as before. And *P*, having two equal and opposite velocities, will be at rest. Now, if a body fastened to a string, revolve round a fixed point *P*, it will revolve uniformly. Hence, the angular motion of *PQ* round *P* when fixed, will be uniform; and therefore it will be uniform when *P* moves uniformly along *AX*.

Hence, the motion of *Q* arises from a uniform angular motion round *P* while *P* moves uniformly in a straight line: and hence its path will be a cycloid, or a trochoid, within or without the cycloid, (see Mr Peacock's Examples, p. 186). Take a point *R* in the radius *PQ*, produced if necessary, such that the velocity of *R* round *P* at rest, may be the same as the velocity of *P* along *AX*. Let a circle be described with radius *PR*, and let *NO*, parallel to *AX*, be a

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really solve the problem in the text, the name of *Tractory*, which is analogous to the names of other curves which occur in Mechanics.

If the Directrix be a circle, and the friction be such as immediately to destroy the motion of the body, the body will move in a curve, the tangent of which, intercepted by a circle, is a constant quantity. This curve is sometimes called the *Complicated Tractrix*. See Cotes, *Harm. Mens.*

For the Tractory, when the directrix is any given curve, see Euler, in the *Nova Acta Acad. Petrop.* 1784. He has there also considered the problem, taking a finite friction into their account: and likewise, what he calls *Compound Tractories*, where there are more points than one attached to the string.

tangent to it. Then, if the circle *RO* roll uniformly along the line *NO*, the point *Q* will trace out a trochoid, which will be the tractory in question. For, in this case it is manifest that *P* the centre of the circle moves uniformly along the line *AX*, and that *PQ* revolves uniformly round *P*; which are the conditions requisite.

The curve will be a cycloid if *R* coincide with *Q*; a trochoid or *prolate* cycloid if *R* be without *Q*; and a *curtate* cycloid if *R* be within *Q*.

If *BC* be the original position of *PQ*, when *Q* is at rest, and if the point *P* begin to move along *AX*, *BC* will be a tangent to the tractory. The curve will be a trochoid of the first kind, and *C* will be its point of contrary flexure. If *BC* be perpendicular to *AB*, the curve will be a cycloid.

PROB. VIII. *Supposing the same things, and that* PQ *does not move in the plane* APQ; *to find the tractory.*

By reasoning similar to that of last problem, it will appear that the motion of *Q* will be determined by supposing *P* to move uniformly along the directrix, and *PQ* to revolve uniformly round *P*: its motion relatively to *P*, being always parallel to a certain fixed plane. Hence, the path of *Q* will be an oblique helix, which may be supposed to be described on the surface of an elliptical cylinder, of which the axis is *AX*.

PROB. IX. *The point* P, *fig.* 16, *moves uniformly in the circumference of a circle* BP, *drawing the point* Q: *to find the tractory described by* Q.

The motion is supposed to be in the plane of the circle *BP*.

Let *AX* be any fixed line; AM, $MP = x$, y, and AN, $NQ = x'$, y', rectangular co-ordinates to *P* and *Q*. The tension of $PQ = p$, and the mass of $Q = m$. Also, let t be the time; and *BP*, proportional to the time, $= nt$, $QPO = \phi$, *PO* being parallel to *AX*, $AP = a$, $PQ = b$. Hence, by resolving the forces,

$$\frac{d^2x'}{dt^2} = -\frac{pg}{m}\cos.\phi,\quad \frac{d^2y'}{dt^2} = -\frac{pg}{m}.\sin.\phi,$$

$$x = a\cos.nt,\quad y = a\sin.nt;$$

$$\therefore\ \frac{d^2x}{dt^2} = -an^2\cos.nt,\quad \frac{d^2y}{dt^2} = -an^2\sin.nt;$$

$$\therefore\ b\frac{d^2\cos.\phi}{dt^2} = \frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2} = -\frac{pg}{m}\cos.\phi + an^2\cos.nt,$$

$$b\frac{d^2\sin.\phi}{dt^2} = \frac{d^2y'}{dt^2} - \frac{d^2y}{dt^2} = -\frac{pg}{m}\sin.\phi + an^2\sin.nt;$$

$$\therefore\ b\sin.\phi\frac{d^2\cos.\phi}{dt^2} - b\cos.\phi\frac{d^2\sin.\phi}{dt^2}$$

$$= an^2(\sin.\phi\cos.nt - \cos.\phi\sin.nt);$$

$$\therefore\ \frac{bd^2\phi}{dt^2} = -an^2\sin.(\phi - nt).$$

Let $\phi - nt = \psi$; then $\frac{d^2\phi}{dt^2} = \frac{d^2\psi}{dt^2}$; and substituting and integrating, we have

$$\frac{d\psi^2}{dt^2} = \frac{2an^2}{b}(C - \cos.\psi).$$

The angle $\phi - nt$ is $QPO - RPO = QPR$. The angular velocity of *PQ* will be greatest when $\phi - nt = \pi$, or when *PQ* coincides with *PA* in direction, as at *P′Q′*. If *C* be less than unity, *PQ* will oscillate about *PA*, while *P* moves uniformly in the circle. If *C* be greater than unity, *Q* will revolve about *P* with a variable velocity, while *P* revolves about *A* uniformly.

The least velocity will be when $\phi - nt = 0$, or when AP, PQ are in a straight line, as at Q''.

PROB. X. *Let* P *move uniformly along a given straight line, while* Q *is drawn along, and also acted on by gravity: to find the motion of* Q.

By reasoning as in Prob. VI, it will appear that the motion of Q with respect to P, will be the same as if P were fixed. Consequently, if PQ move in the same plane, it will be the motion of a circular pendulum, and if not, it will be the *turbinatory* motion of a point in a sphere. This, combined with the rectilinear motion of the point P, will give the actual motion of Q.

Sect. IV. COMPLEX PENDULUMS.

163. PROB. XI. P, Q, *fig.* 17, *are two bodies, of which the first hangs from a fixed point, and the second from the first, by means of inextensible strings* AP, PQ: *it is required to determine the small oscillations.*

Let $AM = x$, $MP = y$, $AN = x_1$, $NQ = y_1$, $AP = a$, $PQ = a_1$. Mass of $P = m$, of $Q = m_1$; tension of $AP = p$, of $PQ = p_1$.

Hence, resolving the forces p, p_1, we have

$$\left.\begin{aligned} \frac{d^2y}{dt^2} &= \frac{p_1 g}{m} \cdot \frac{y_1 - y}{a_1} - \frac{pg}{m} \cdot \frac{y}{a} \\ \frac{d^2y_1}{dt^2} &= -\frac{p_1 g}{m_1} \cdot \frac{y_1 - y}{a_1} \end{aligned}\right\} \ldots\ldots\ldots (1).$$

By combining these with the equations in x, x_1, and with the two

$$x^2 + y^2 = a^2, \quad (x_1 - x)^2 + (y_1 - y)^2 = a_1^2;$$

we should, by eliminating p, p^1, find the motion. But when the oscillations are small, we may approximate in a more simple manner.

Let β, β_1 be the initial values of y, y_1. Then manifestly, p, p_1 will depend on the initial position of the bodies, and on their position at the time t: and hence we may suppose

$$p = M + P\beta + Q\beta_1 + Ry + Sy_1 + \&c.: \text{ and similarly for } p_1.$$

Now, in the equations of motion above, p, p_1, are multiplied by y, $y_1 - y$, which, since the oscillations are very small, are also very small quantities (viz. of the order β). Hence, their products with β will be of the order β^2, and may be neglected, and we may suppose p reduced to its first term M.

M is the tension of AP, when β, β_1, &c. are all $= 0$. Hence, it is the tension when P, Q hang at rest from A, and consequently, $M = m + m_1$; similarly, the first term of p_1, is m_1, and this may be put for p_1. Substituting these values, and dividing by g, equations (1) become

$$\left.\begin{aligned} \frac{1}{g}\frac{d^2y}{dt^2} &= -\left(\frac{m_1}{ma_1}+\frac{m+m_1}{ma}\right)y + \frac{m_1}{ma_1}\cdot y_1 \\ \frac{1}{g}\frac{d^2y_1}{dt^2} &= \frac{y}{a_1}-\frac{y_1}{a_1} \end{aligned}\right\}\dots\dots\dots(2).$$

Multiply the second of these equations by λ, and add it to the first, and we have

$$\frac{1}{g}\left(\frac{d^2y}{dt^2}+\lambda\frac{d^2y_1}{dt^2}\right) = -\left(\frac{m_1}{ma_1}+\frac{m+m_1}{ma}-\frac{\lambda}{a_1}\right)y - \left(\frac{\lambda}{a_1}-\frac{m_1}{ma_1}\right)y_1:$$

and manifestly this can be solved, if the second side can be put in the form $-k(y+\lambda y_1)$; that is, if

$$k = \frac{m_1}{ma_1}+\frac{m+m_1}{ma}-\frac{\lambda}{a_1};$$

$$k\lambda = \frac{\lambda}{a_1}-\frac{m_1}{ma_1};$$

$$\left.\begin{aligned} \text{or } a_1k &= \frac{m_1}{m}+\frac{a_1}{a}+\frac{m_1a_1}{ma}-\lambda \\ -\frac{m_1}{m} &= (a_1k-1)\lambda \end{aligned}\right\}\dots\dots\dots(3).$$

Eliminating λ, we have

$$(a_1 k - 1)\, a_1 k - \frac{m_1}{m} = (a_1 k - 1)\left(\frac{m_1}{m} + \frac{a_1}{a} + \frac{m_1 a_1}{ma}\right).$$

Hence, $$(a_1 k)^2 - \left(1 + \frac{m_1}{m}\right)\left(1 + \frac{a_1}{a}\right) a_1 k = -\frac{a_1}{a} - \frac{m_1 a_1}{ma}\,.\ (4).$$

From this equation we obtain two values of k. Let these be denoted by ${}^1k$, ${}^2k$; and let the corresponding values of λ be ${}^1\lambda$, ${}^2\lambda$. Hence, we have these equations

$$\frac{1}{g}\left(\frac{d^2y}{dt^2} + {}^1\lambda\frac{d^2y}{dt^2}\right) = -\,{}^1k\,(y + {}^1\lambda y_1),$$

$$\frac{1}{g}\left(\frac{d^2y}{dt^2} + {}^2\lambda\frac{d^2y}{dt^2}\right) = -\,{}^2k\,(y + {}^2\lambda y_1).$$

And it is easily seen, as in Article 20, Ex. 2, that the integrals of these equations are

$$y + {}^1\lambda y_1 = {}^1C \cos.\, t\sqrt{({}^1kg)} + {}^1D \sin.\, t\sqrt{({}^1kg)},$$

$$y + {}^2\lambda y_1 = {}^2C \cos.\, t\sqrt{({}^2kg)} + {}^2D \sin.\, t\sqrt{({}^2kg)}.$$

${}^1C$, ${}^1D$, ${}^2C$, ${}^2D$ being arbitrary constants. But we may suppose

$${}^1C = {}^1E \cos.\, {}^1\epsilon,\quad {}^1D = {}^1E \sin.\, {}^1\epsilon,\quad {}^2C = {}^2E \cos.\, {}^2\epsilon,\quad {}^2D = {}^2E \sin.\, {}^2\epsilon,$$

where ${}^1E$, ${}^2E$, ${}^1\epsilon$, ${}^2\epsilon$, are other arbitrary constants. By introducing these values, we find

$$\left.\begin{aligned} y + {}^1\lambda y_1 &= {}^1E \cos.\,\left\{t\sqrt{({}^1kg)} + {}^1\epsilon\right\} \\ y + {}^2\lambda y_1 &= {}^2E \cos.\,\left\{t\sqrt{({}^2kg)} + {}^2\epsilon\right\} \end{aligned}\right\}\cdots\cdots\cdots(5).$$

From these we easily find

$$\left.\begin{aligned} y &= \frac{{}^2\lambda\, {}^1E}{{}^2\lambda - {}^1\lambda}\cdot\cos.\left\{t\sqrt{({}^1kg)} + {}^1\epsilon\right\} + \frac{{}^1\lambda\, {}^2E}{{}^2\lambda - {}^1\lambda}\cos.\left\{t\sqrt{({}^2kg)} + {}^2\epsilon\right\} \\ y_1 &= \frac{{}^1E}{{}^1\lambda - {}^2\lambda}\cdot\cos.\left\{t\sqrt{({}^1kg)} + {}^1\epsilon\right\} + \frac{{}^2E}{{}^1\lambda - {}^2\lambda}\cos.\left\{t\sqrt{({}^2kg)} + {}^2\epsilon\right\} \end{aligned}\right\}(6).$$

The arbitrary quantities ${}^1E$, ${}^1\epsilon$, &c. depend on the initial position and velocity of the points. If the velocities of P, Q be $= 0$, when $t = 0$, we shall have ${}^1\epsilon$, ${}^2\epsilon$ each $= 0$, as appears by taking the differentials of y, y_1.

Cor. 1. If either of the two ${}^1E$, ${}^2E$, be $= 0$, we shall have, (supposing the latter case, and omitting ${}^1\epsilon$)

$$y = \frac{{}^2\lambda\, {}^1E}{{}^2\lambda - {}^1\lambda} \cos. t \sqrt{{}^1kg},$$

$$y_1 = \frac{{}^1E}{{}^1\lambda - {}^2\lambda} \cos. t \sqrt{{}^1kg}.$$

Hence, it appears that the oscillations in this case are *symmetrical:* that is, the bodies P, Q come to the vertical line at the same time (when $t = 0$), have similar and equal motions on the two sides of it, and reach their greatest distances from it at the same time. It is easy to see that in this case, the motion has the same law of time and velocity as the motion in a cycloidal pendulum; and the time of a semi-oscillation, in this case, extends from when

$$t = 0 \text{ to when } t\sqrt{({}^1kg)} = \pi, \text{ or } t = \frac{\pi}{\sqrt{({}^1kg)}}.$$

Also if β, β_1 be the greatest horizontal deviation of P, Q, we shall have

$$y = \beta \cos. t \sqrt{({}^1kg)}, \quad y_1 = \beta_1 \cos. t \sqrt{({}^1kg)}.$$

In order to find the original relation of β, β_1*, that the oscillations may be of this kind (the original velocities being 0), we must have, by equation (5), since ${}^2E = 0$,

$$\beta + {}^2\lambda \beta_1 = 0.$$

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* The ocillations will be symmetrical if the forces which urge P and Q to the vertical, be as PM, QN, as is easily seen. Hence, the conditions for symmetrical oscillation might be determined by finding what must be the position of P, Q, in order that this might originally be the relation of the forces.

Similarly, if we had $\beta + {}^1\lambda\beta_1 = 0$, we should have ${}^1E = 0$, and the oscillations would be symmetrical, and would employ a time

$$\frac{\pi}{\sqrt{({}^2kg)}}.$$

Cor. 2. When neither of these relations obtains, the oscillations may be considered as compounded of two, in the following manner. Suppose that we put

$$y = H\cos.t\sqrt{({}^1kg)} + K\cos.t\sqrt{({}^2kg)}\ldots\ldots(7),$$

omitting ${}^1\epsilon$, ${}^2\epsilon$, and altering the constants in equations (6); and suppose that in fig. 17, we take $Mp = H\cos.t\sqrt{({}^1kg)}$. Then p will oscillate about M, according to the law of a cycloidal pendulum (neglecting the vertical motion). Also pP will $= K\cos.t\sqrt{({}^2kg)}$. Hence, P oscillates about p according to a similar law, while p oscillates about M. And in the same way, we may have a point q so moved, that Q shall oscillate about q in a time $\frac{\pi}{\sqrt{({}^2kg)}}$, while q oscillates about N in a time $\frac{\pi}{\sqrt{({}^1kg)}}$. And hence, the motion of the pendulum APQ is compounded of the motion of Apq oscillating symmetrically about the vertical line, and of APQ oscillating symmetrically about Apq, as if that were a fixed vertical line.

When a pendulum oscillates in this manner, it will never return exactly to its original position, if $\sqrt{{}^1k}$ and $\sqrt{{}^2k}$ are incommensurable. If $\sqrt{{}^1k}$ and $\sqrt{{}^2k}$ are commensurable, so that we have $m\sqrt{{}^1k} = n\sqrt{{}^2k}$, m and n being whole numbers, the pendulum will at certain intervals, return to its original position. For let

$$t\sqrt{({}^1kg)} = 2n\pi;\ \text{then } t\sqrt{({}^2kg)} \text{ will } = 2m\pi;\ \text{and by (7)},$$

$$y = H\cos.2n\pi + K\cos.2m\pi = H + K,$$

which is the same as when $t = 0$. And similarly, after an interval such that $t\sqrt{({}^1kg)} = 4n\pi$, $6n\pi$, &c. the pendulum will return to its original position, having described in the intermediate times, similar cycles of oscillations.

Ex. Let $m_1 = m$, and $a_1 = a$, to determine the oscillations.

Here equation (4) becomes,

$$a^2 k^2 - 4ak = -2, \quad ak = 2 \pm \sqrt{2}.$$

Also, by equation (3),

$$ak = 3 - \lambda\,; \quad \therefore\ {}^1\lambda = 1 + \sqrt{2}\,; \quad {}^2\lambda = 1 - \sqrt{2}.$$

Hence, in order that the oscillations may be symmetrical, we must either have

$$\beta + (1 + \sqrt{2})\,\beta_1 = 0\,; \text{ whence } \beta_1 = -(\sqrt{2} - 1)\,\beta\,;$$
$$\text{or } \beta - (\sqrt{2} - 1)\,\beta_1 = 0\,; \text{ whence } \beta_1 = (\sqrt{2} + 1)\,\beta.$$

The two arrangements indicated by these equations are represented, fig. 18, and fig. 19. The first corresponds to

$$\beta_1 = (\sqrt{2} + 1)\,\beta, \text{ or } QN = (\sqrt{2} + 1)\,PM.$$

In this case, the pendulum will oscillate into the position $AP'Q'$, similarly situated on the other side of the line; and the time of this complete oscillation will be

$$\frac{\pi}{\sqrt{\left\{\frac{g}{a}(2 - \sqrt{2})\right\}}} = \frac{\pi}{\sqrt{(2 - \sqrt{2})}}\sqrt{\frac{a}{g}}.$$

In the other case, corresponding to $\beta_1 = -(\sqrt{2} - 1)\,\beta$, Q is on the other side of the vertical line, and $QN = (\sqrt{2} - 1)\,PM$. The pendulum oscillates into the position $AP'Q'$, the point O remaining always in the vertical line; and the time of an oscillation is

$$\frac{\pi}{\sqrt{(2 + \sqrt{2})}}\sqrt{\frac{a}{g}}.$$

The lengths of simple pendulums which would oscillate respectively in these times, would be

$$\frac{a}{2 - \sqrt{2}}, \text{ and } \frac{a}{2 + \sqrt{2}}, \text{ or } 1.707\,a \text{ and } .293\,a.$$

If neither of these arrangements exist originally, let β, β_1, be the original values of y, y_1, when t is 0. Then making $t=0$ in equations (5), we have

$$^1E = \beta + (\sqrt{2}+1)\beta_1, \quad ^2E = \beta - (\sqrt{2}-1)\beta_1.$$

And these being known, we have the motion by equations (6).

Prob. XII. *Any number of material points* $P_1\, P_2\, P_3 \ldots Q$, *fig.* 20, *hang, by means of a string without weight, from a point* A: *it is required to determine their small oscillations in a vertical plane.*

Let AN be a vertical abscissa, and P_1M_1, P_2M_2, &c. horizontal ordinates; so that

$$AM_1 = x_1, \quad AM_2 = x_2, \quad AM_3 = x_3, \quad \text{\&c.}$$

$$P_1M_1 = y_1, \quad P_2M_2 = y_2, \quad P_3M_3 = y_3, \quad \text{\&c.}$$

$$AP_1 = a_1, \quad P_1P_2 = a_2, \quad P_2P_3 = a_3, \quad \text{\&c.}$$

$$\text{tension of } AP_1 = p_1, \text{ of } P_1P_2 = p_2, \text{ of } P_2P_3 = p_3, \text{ \&c.}$$

$$\text{mass of } P_1 = m_1, \text{ of } P_2 = m_2, \text{ of } P_3 = m_3, \text{ \&c.}$$

Hence, we have these equations, by resolving the forces parallel to the horizon,

$$\left.\begin{array}{l}
\dfrac{d^2y_1}{dt^2} = -\dfrac{p_1g}{m_1}\cdot\dfrac{y_1}{a_1} + \dfrac{p_2g}{m_1}\cdot\dfrac{y_2-y_1}{a_2} \\[2ex]
\dfrac{d^2y_2}{dt^2} = -\dfrac{p_2g}{m_2}\cdot\dfrac{y_2-y_1}{a_2} + \dfrac{p_3g}{m_2}\cdot\dfrac{y_3-y_2}{a_3} \\[2ex]
\dfrac{d^2y_3}{dt^2} = -\dfrac{p_3g}{m_3}\cdot\dfrac{y_3-y_2}{a_3} + \dfrac{p_4g}{m_3}\cdot\dfrac{y_4-y_3}{a_4} \\[2ex]
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[2ex]
\dfrac{d^2y_n}{dt^2} = -\dfrac{p_ng}{m_n}\cdot\dfrac{y_n-y_{n-1}}{a_n}
\end{array}\right\} \cdots\cdots\cdots(1).$$

And, as in the last problem, it will appear that p_1, p_2, &c. may, for these small oscillations, be considered as constant, and of the same value as in the state of rest. Hence, if

$$m_1 + m_2 + m_3 \ldots + m_n = M,$$

$$p_1 = M,\quad p_2 = M - m_1,\quad p_3 = M - m_1 - m_2,\quad \text{\&c.}$$

Also dividing by g, the above equations may be put in this form

$$\left.\begin{aligned}
\frac{1}{g}\,\frac{d^2y_1}{dt^2} &= -\left(\frac{p_1}{m_1a_1} + \frac{p_2}{m_1a_2}\right)y_1 + \frac{p_2y_2}{m_1a_2}\\
\frac{1}{g}\,\frac{d^2y_2}{dt^2} &= \frac{p_2y_1}{m_2a_2} - \left(\frac{p_2}{m_2a_2} + \frac{p_3}{m_2a_3}\right)\cdot y_2 + \frac{p_3y_3}{m_2a_3}\\
\frac{1}{g}\,\frac{d^2y_3}{dt^2} &= \frac{p_3y_2}{m_3a_3} - \left(\frac{p_3}{m_3a_3} + \frac{p_4}{m_3a_4}\right)y_3 + \frac{p_4y_4}{m_3a_4}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{1}{g}\,\frac{d^2y_n}{dt^2} &= \frac{p_ny_{n-1}}{m_na_n} - \frac{p_ny_n}{m_na_n}
\end{aligned}\right\}\ldots\ldots(2).$$

The first and last equations are seen to be symmetrical with the rest if we observe that $y_0 = 0$, and $p_{n+1} = 0$.

Now, if we multiply these equations respectively by 1, λ, λ', λ'', &c. and add them, we have $\dfrac{1}{g}\,\dfrac{d^2(y_1 + \lambda y_2 + \lambda' y_3 + \text{\&c.})}{dt^2}$

$$\begin{aligned}
&= \left\{-\frac{p_1}{m_1a_1} - \frac{p_2}{m_1a_2} + \frac{\lambda p_2}{m_2a_2}\right\}y_1\\
&+ \left\{\frac{p_2}{m_1a_2} - \lambda\left(\frac{p_2}{m_2a_2} + \frac{p_3}{m_2a_3}\right) + \frac{\lambda' p_3}{m_3a_3}\right\}y_2\\
&+ \left\{\frac{\lambda p_3}{m_2a_3} - \lambda'\left(\frac{p_3}{m_3a_3} + \frac{p_4}{m_3a_4}\right) + \frac{\lambda'' p_4}{m_4a_4}\right\}y_3\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+ \left\{\frac{\lambda'^{(n-3)}p_n}{m_{n-1}a_n} - \frac{\lambda'^{(n-2)}p_n}{m_na_n}\right\}y_n.
\end{aligned}$$

And this will be integrable, if the right-hand side of the equation be reducible to this form

$$-k\,(y_1+\lambda y_2+\lambda' y_3+\&c.).$$

That is, if

$$\left.\begin{aligned}
k &= \frac{p_1}{m_1 a_1}+\frac{p_2}{m_1 a_2}-\frac{\lambda p_2}{m_2 a_2}\\
k\lambda &= -\frac{p_2}{m_1 a_2}+\lambda\left(\frac{p_2}{m_2 a_2}+\frac{p_3}{m_2 a_3}\right)-\frac{\lambda' p_3}{m_3 a_3}\\
k\lambda' &= -\frac{\lambda p_3}{m_2 a_3}+\lambda'\left(\frac{p_3}{m_3 a_3}+\frac{p_4}{m_3 a_4}\right)+\frac{\lambda'' p_4}{m_4 a_4}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
k\lambda'^{(n-2)} &= -\frac{\lambda'^{(n-3)} p_n}{m_{n-1} a_n}+\frac{\lambda'^{(n-2)} p_n}{m_n a_n}
\end{aligned}\right\}\ldots\ldots(3).$$

If we now eliminate λ, λ', λ'', &c. from these n equations, it is easily seen that we shall have an equation of n dimensions in k. Let ${}^1k$, ${}^2k$, ${}^3k\ldots{}^nk$ be the n values of k, deduced from this equation; then, for each of these values there is a value of λ', λ'', λ''' easily deduced from equations (3), which values we may represent by ${}^1\lambda$, ${}^1\lambda'$, ${}^1\lambda''$, ${}^1\lambda'''$, &c., ${}^2\lambda'$, ${}^2\lambda''$, ${}^2\lambda'''$, &c. Hence, we have these equations, by taking corresponding values of λ and k,

$$\frac{1}{g}\,\frac{d^2(y_1+{}^1\lambda y_2+{}^1\lambda' y_3+\&c.)}{dt^2}=-\,{}^1k\,.\,(y_1+{}^1\lambda y_2+{}^1\lambda' y_3+\&c.)$$

$$\frac{1}{g}\,\frac{d^2(y_1+{}^2\lambda y_2+{}^2\lambda' y_3+\&c.)}{dt^2}=-\,{}^2k\,\,(y_2+{}^2\lambda y_2+{}^2\lambda' y_3+\&c.)$$

and so on, making n equations.

Integrating each of these equations we get, as in the last problem,

$$\left.\begin{aligned}
y_1+{}^1\lambda y_2+{}^1\lambda' y_3+\&c. &= {}^1E\cos.\left\{t\sqrt{({}^1kg)}+{}^1\epsilon\right\}\\
y_1+{}^2\lambda y_2+{}^2\lambda' y_3+\&c. &= {}^2E\cos.\left\{t\sqrt{({}^2kg)}+{}^2\epsilon\right\}\\
\&c. \qquad &= \qquad \&c.
\end{aligned}\right\}\ldots\ldots(4).$$

${}^1E$, ${}^2E$, &c. ${}^1\epsilon$, ${}^2\epsilon$, &c. being arbitrary constants.

From these n simple equations, we can, without difficulty, obtain the n quantities y_1, y_2, &c. And it is manifest that the results will be of this form

$$\left.\begin{aligned} y_1 &= {}^1H_1 \cos.\{t\sqrt{({}^1kg)}+{}^1\epsilon\} + {}^2H_1 \cos.\{t\sqrt{({}^2kg)}+{}^2\epsilon\} + \&c. \\ y_2 &= {}^1H_2 \cos.\{t\sqrt{({}^1kg)}+{}^1\epsilon\} + {}^2H_2 \cos.\{t\sqrt{({}^2kg)}+{}^2\epsilon\} + \&c. \\ \&c. &= \&c. \end{aligned}\right\} (5),$$

where ${}^1H_1$, ${}^1H_2$, &c. must be deduced from β_1, β_2, &c. the original values of y_1, y_2, &c.

If the points have no velocities at first, (i. e. when $t=0$) we shall have ${}^1\epsilon = 0$, ${}^2\epsilon = 0$, &c.

COR. 1. We may have symmetrical oscillations in the following manner. If, of the quantities ${}^1E$, ${}^2E$, ${}^3E$, &c. all vanish except one, for instance, ${}^nE$; we have

$$\left.\begin{aligned} &y_1 + {}^1\lambda y_2 + {}^1\lambda' y_3 + \&c. = 0, \\ &y_1 + {}^2\lambda y_2 + {}^2\lambda' y_3 + \&c. = 0, \\ &y_1 + {}^3\lambda y_2 + {}^3\lambda' y_3 + \&c. = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &y_1 + {}^n\lambda y_2 + {}^n\lambda' y_3 + \&c. = {}^nE \cos. t\sqrt{({}^nkg)}, \text{ omitting } {}^n\epsilon \end{aligned}\right\} \ldots (6).$$

From the first $n-1$ of these equations, it appears that y_2, y_3, &c. are in a given ratio to y_1; and hence,

$$y_1 + {}^n\lambda y_2 + {}^n\lambda' y_3 + \&c.$$

is a given multiple of y_1 and $= my_1$ suppose. Hence, we have

$$my_1 = {}^nE \cos. \sqrt{({}^nkg)};$$

or, omitting the index n, which is now unnecessary,

$$my_1 = E \cos. t\sqrt{(kg)}. \quad \text{Also, if } y_2 = e_2 y_1,$$

$$my_1 = Ee_2 \cos. t\sqrt{(kg)}, \text{ and similarly for } y_3, \&c.$$

Hence, it appears that in this case the oscillations are *symmetrical.* All the points come into the vertical line at the same time, and move similarly and contemporaneously on the two sides of it. The relation among the original ordinates β_1, β_2, β_3, &c. which must subsist in order that the oscillations may be of this kind, is given by the $n-1$ equations (8),

$$\begin{aligned}\beta_1 + {}^1\lambda\beta_2 + {}^1\lambda'\beta_3 + \&c. &= 0,\\ \beta_1 + {}^2\lambda\beta_2 + {}^2\lambda'\beta_3 + \&c. &= 0,\\ \beta_1 + {}^3\lambda\beta_2 + {}^3\lambda'\beta_3 + \&c. &= 0,\\ \&c. &= \&c.\end{aligned}$$

These give the *proportion* of β_1, β_2, &c.; the arbitrary constant ${}^nE$, in the remaining equation, gives the *actual quantity* of the original displacement.

Also, we may take any one of the quantities ${}^1E$, ${}^2E$, ${}^3E$, &c. for that which does not vanish; and hence obtain, in a different way, such a system of $n-1$ equations as has just been described. Hence, there are n different relations among β_1, β_2, &c., or n different modes of arrangement, in which the points may be placed, so as to oscillate symmetrically*.

The time of oscillation in each of these arrangements is easily known; the equation

$$my_1 = {}^nE \cos. t\sqrt{({}^nkg)},$$

shews that an oscillation employs a time $t = \dfrac{\pi}{\sqrt{({}^nkg)}}$. And hence, if all the roots ${}^1k$, ${}^2k$, ${}^3k$, &c. be different, the time is different for each different arrangement.

COR. 2. If the initial arrangement of the points be different from all those thus obtained, the oscillations of the pendulum may always be considered as compounded of n symmetrical oscillations. That is, if an imaginary pendulum oscillate symmetrically about the vertical line in a time $\frac{\pi}{\sqrt{({}^1kg)}}$; and a second imaginary pendulum oscillate about the place of the first, considered as a fixed line, in the time $\frac{\pi}{\sqrt{({}^2kg)}}$; and a third about the second, in the same manner, in the time $\frac{\pi}{\sqrt{({}^3kg)}}$; and so on; the n^{th} pendulum may always be made to coincide perpetually with the real pendulum, by properly adjusting the amplitudes of the imaginary oscillations. This appears by considering the equations (6),

---

* We might here also find these positions, which give symmetrical oscillations, by requiring the force in each of the ordinates P_1M_1, P_2M_2, to be as the distance; in which case the points P_1, P_2, &c. would all come to the vertical at the same time.

If the quantities $\sqrt{{}^1k}$, $\sqrt{{}^2k}$, &c. have one common measure, there will be a time after which the pendulum will come into its original position. And it will describe similar successive cycles of vibrations. If these quantities be not commensurable, no portion of its motion will be similar to any preceding portion.

$$y_1 = {}^1H_1 \cos. t\sqrt{({}^1kg)} + {}^2H_1 \cos. t\sqrt{({}^2kg)} + \&c.$$
$$\&c. = \&c.$$

This principle of the *co-existence of vibrations* is applicable in all cases where the vibrations are indefinitely small. In all such cases each set of symmetrical vibrations takes place, and affects the system as if that were the only motion which it experienced.

A familiar instance of this principle is seen in the manner in which the circular vibrations, produced by dropping stones into still water, spread from their respective centres, and cross without disfiguring each other.

If the oscillations be not all made in one vertical plane, we may take a horizontal ordinate z perpendicular to y. The oscillations in the direction of y will be the same as before, and there will be similar results obtained with respect to the oscillations in the direction of z.

We have supposed that the motion in the direction of x, the vertical axis, may be neglected, which is true when the oscillations are very small.

Ex. Let there be three bodies all equal; (each $= m$), and also their distances a_1, a_2, a_3, all equal; (each $= a$).

Here $p_1 = 3m$, $p_2 = 2m$, $p_3 = m$, and equations (3) become

$$\begin{aligned} ak &= 5 - 2\lambda, \\ ak\lambda &= -2 + 3\lambda - \lambda', \\ ak\lambda' &= -\lambda + \lambda'. \end{aligned}$$

Eliminating k, we have

$$\begin{aligned} 5\lambda - 2\lambda^2 &= -2 + 3\lambda - \lambda', \\ 5\lambda' - 2\lambda\lambda' &= -\lambda + \lambda', \end{aligned}$$

$$\text{or,}\quad \lambda' = 2\lambda^2 - 2\lambda - 2,$$

$$4\lambda' - 2\lambda\lambda' = -\lambda, \quad \lambda' = \frac{\lambda}{2\lambda - 4};$$

$$\therefore\ (2\lambda^2 - 2\lambda - 2)(2\lambda - 4) = \lambda,$$

$$\text{or,}\quad \lambda^3 - 3\lambda^2 + \tfrac{3}{4}\lambda + 2 = 0,$$

which may be solved by Trigonometrical Tables. We shall find three values of λ.

Hence, we have a value of λ' corresponding to each value of λ; and then by equations (8),

$$\left.\begin{aligned} \beta_1 + {}^1\lambda\beta_2 + {}^1\lambda'\beta_3 = 0 \\ \beta_1 + {}^2\lambda\beta_2 + {}^2\lambda'\beta_3 = 0 \end{aligned}\right\} \cdots\cdots\cdots (8').$$

Whence we find β_2 and β_3 in terms of β_1.

We shall thus find* $\beta_2 = 2.295\,\beta_1$,

$$\text{or } \beta_2 = 1.348\,\beta_1,$$

$$\text{or } \beta_2 = -.643\,\beta_1,$$

according as we take the different values of λ.

And the times of oscillation in each case will be found by taking the value of $ak = 5 - 2\lambda$; that value of λ being taken which is not used in equation (8′). For the time of oscillation will be given by making $t\sqrt{(kg)} = \pi$.

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* Euler, *Com. Petrop.* Tom. VIII, p. 37.

If the values of β_1, β_2, β_3 have not this initial relation, the oscillations will be compounded in a manner similar to that described, p. 75, in the Example for two bodies only.

Prob. XIII. *A flexible chain, of uniform thickness, hangs from a fixed point: to find its initial form, that its small oscillations may be symmetrical.*

Let, in fig. 21, AM, the vertical abscissa $= x$; MP, the horizontal ordinate $= y$; $AP = s$, and the whole length $AC = a$; $\therefore AP = a - s$. And, in the same way as in Prob. XI, the tension at P will, when the oscillations are small, be the weight of PC, and may be represented by $a - s$. This tension will act in the direction of a tangent at P, and hence, the part of it in the direction PM will be *tension* $\times \dfrac{dy}{ds}$, or $(a-s)\dfrac{dy}{ds}$.

Now, if we take any portion $PQ = h$, we shall find the horizontal force at Q in the same manner. For the point Q, supposing s the independent variable,

$$\frac{dy}{ds} \text{ becomes } \frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \frac{d^3y}{ds^3}\cdot\frac{h^2}{1\,.\,2} + \&c.$$

Also, the tension will be $(a - s - h)$. Hence, the horizontal force in direction NQ, is

$$(a-s-h)\left(\frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \frac{d^3y}{ds^3}\,\frac{h^2}{1\,.\,2},\ \&c.\right)$$

Subtracting from this the force in PM, we have the force on PQ horizontally

$$= (a-s)\left(\frac{d^2y}{ds^2}\cdot\frac{h}{1} + \frac{d^3y}{ds^3}\cdot\frac{h^2}{1\,.\,2} + \&c.\right)$$

$$- h\left(\frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \&c.\right)$$

And the mass of PQ being represented by h, the accelerating force $\left(= \dfrac{\text{pressure}\,.\,g}{\text{mass}}\right)$ is found. But since the different points of *PQ* move with different velocities, this expression is only applicable when h is indefinitely small. Hence, supposing Q to approach to, and coincide with P, we have, when h vanishes,

$$\text{accelerating force on } P = (a - s)\frac{d^2y}{ds^2} - \frac{dy}{ds}.$$

But, since the oscillations are indefinitely small, x coincides with s, and we have

$$\text{accelerating force on } P = (a - x)\frac{d^2y}{dx^2} - \frac{dy}{dx}.$$

Now, in order that the oscillations may be symmetrical, this force must be in the direction *PM*, and proportional to *PM*, in which case all the points of *AC* will come to the vertical *AB* at once. Hence, we must have

$$(a - x)\frac{d^2y}{dx^2} - \frac{dy}{dx} = -ky \ldots\ldots\ldots (1),$$

k being some constant quantity to be determined.

This equation cannot be integrated in finite terms. To obtain a series, let

$$y = A + B(a - x) + C(a - x)^2 + D(a - x)^3 + \&c.$$

$$\frac{dy}{dx} = -B - 2C(a - x) - 3D(a - x)^2 + \&c.$$

$$\frac{d^2y}{dx^2} = 1 . 2C + 2 . 3D(a - x) + \&c.$$

Hence, $0 = (a - x)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + ky$; gives

$$\begin{aligned} 0 &= 1 . 2C(a - x) + 2 . 3D(a - x)^2 + \&c. \\ &+ B + 2C(a - x) + 3D(a - x)^2 + \&c. \\ &+ kA + kB(a - x) + kC(a - x)^2 + \&c. \end{aligned}$$

Equating coefficients, we have

$$B = -kA,\quad 2^2 . C = -kB,\quad 3^2 D = -kC,\quad \&c.$$

$$\therefore\ B = -kA, \qquad C = \frac{k^2 A}{2^2}, \qquad D = -\frac{k^3 A}{2^2 . 3^2}, \text{ \&c.}$$

and $y = A\left\{1 - k(a-x) + \frac{k^2}{2^2}(a-x)^2 - \frac{k^3}{2^2 3^2}(a-x)^3 + \text{\&c.}\right\} \ldots (2)$.

Here A is BC, the value of y when $x = a$.

When $x = 0$, $y = 0$; $\therefore\ 1 - ka + \frac{k^2 a^2}{2^2} - \frac{k^3 a^3}{2^2 3^2} + \text{\&c.} = 0 \ldots (3)$.

From this equation k is to be found. The equation has an infinite number of dimensions, and hence k will have an infinite number of values, which we may call ${}^1k, {}^2k, {}^3k, \ldots {}^nk \ldots$; and these give an infinite number of initial forms, for which the chain may perform symmetrical oscillations.

The time of oscillation for each of these forms will be found thus. At the distance y, the force is kgy; hence, by Art. 12. Ex. 1.

time to the vertical $= \frac{\pi}{2\sqrt{(kg)}}$; and time of oscillation $= \frac{\pi}{\sqrt{(kg)}}$*.

The points where the curve cuts the axis will be found by putting $y = 0$. Hence, taking the value ${}^nk$ of k, we have

$$0 = 1 - {}^nk\,(a-x) + \frac{{}^nk^2\,(a-x)^2}{2^2} - \frac{{}^nk^3\,(a-x)^2}{2^2 . 3^2} + \text{\&c.}$$

which will manifestly be verified, if

${}^nk\,(a-x) = {}^1k a$, or ${}^nk\,(a-x) = {}^2k a$, or ${}^nk\,(a-x) = {}^3k a$, &c.

because ${}^1ka$, ${}^2ka$, &c. are roots of equation (3).

That is, if

$$x = a\left(1 - \frac{{}^1k}{{}^nk}\right), \text{ or } = a\left(1 - \frac{{}^2k}{{}^nk}\right), \text{ or } = a\left(1 - \frac{{}^3k}{{}^nk}\right), \text{ \&c.}$$

---

* The greatest value of ka is about 1.44, (Euler, *Com. Acad. Petrop.* Tom. VIII. p. 43.). And the time of oscillation for this value, is the same as that of a simple pendulum, whose length is $\frac{2}{3}\,a$, nearly.

Suppose ${}^1k$, ${}^2k$, ${}^3k$, &c. to be the roots in the order of their magnitude, ${}^1k$ being the least.

Then, if for ${}^nk$ we take ${}^1k$, all these values of x will be negative, and the curve will never cut the vertical axis below A.

If for ${}^nk$ we take ${}^2k$, all the values of x will be negative except the first; therefore, the curve will cut AB in one point. If we take ${}^3k$, all the values will be negative except the two first, and the curve cuts AB in two points; and so on.

Hence, the forms for which the oscillations will be symmetrical, are of the kind represented in fig. 22. And there are an infinite number of them, each cutting the axis in a different number of points.

If we represent equation (2) in this manner, $y = A\phi(k, x)$; it is evident that $y = {}^1A\phi({}^1k, x)$, $y = {}^2A\phi({}^2k, x)$, &c. will each satisfy equation (1). Hence, as in Prob. XI, if we put

$$y = {}^1A\phi({}^1k, x) + {}^2A\phi({}^2k, x) + \&c.$$

and if ${}^1A$, ${}^2A$, &c. can be so assumed, that this shall represent a given initial form of the chain, its oscillations will be compounded of as many co-existing symmetrical ones, as there are terms ${}^1A$, ${}^2A$, &c.

Sect. V. THE MOTION OF BODIES CONNECTED BY STRINGS.

PROB. XIV. *Two bodies, connected by a string passing over a fixed pully, move on two given curves: to determine their motions.*

The motions are supposed to take place in a vertical plane.

Let A, fig. 23, be the pully, AM a vertical line, BP, CQ, the given curves, and MP, NQ horizontal ordinates to the places of the bodies.

$AM = x$, $MP = y$, $AN = x'$, $NQ = y'$; $AP = r$, $AQ = r'$, curve $BP = s$, $CQ = s'$: mass of $P = m$, of $Q = m'$; re-action of surface $BP = n$, of $CQ = n'$; tension of the string $PAQ = p$.

Then, by resolving the forces which act upon each of the bodies, we shall easily find the equations

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= g - \frac{pg}{m}\cdot\frac{x}{r} - \frac{ng}{m}\cdot\frac{dy}{ds}; \quad \frac{d^2y}{dt^2} = -\frac{pg}{m}\cdot\frac{y}{r} + \frac{ng}{m}\cdot\frac{dx}{ds} \\ \frac{d^2x'}{dt^2} &= g - \frac{p'g}{m'}\cdot\frac{x'}{r'} - \frac{n'g}{m'}\cdot\frac{dy'}{ds'}; \quad \frac{d^2y'}{dt^2} = -\frac{p'g}{m'}\cdot\frac{y'}{r'} + \frac{n'g}{m'}\cdot\frac{dx'}{ds'} \end{aligned}\right\}\ldots(1).$$

Multiply by $2m\frac{dx}{dt}$, $2m\frac{dy}{dt}$, $2m'\frac{dx'}{dt}$, $2m'\frac{dy'}{dt}$, and add the two upper equations, and the two lower ones; and we have

$$2m\left(\frac{dx}{dt}\,\frac{d^2x}{dt^2} + \frac{dy}{dt}\,\frac{d^2y}{dt^2}\right) = 2mg\frac{dx}{dt} - 2pg.\left(\frac{x}{r}\,\frac{dx}{dt} + \frac{y}{r}\,\frac{dy}{dt}\right)$$

$$2m'\left(\frac{dx'}{dt}\,\frac{d^2x'}{dt^2} + \frac{dy'}{dt}\,\frac{d^2y'}{dt^2}\right) = 2m'g\frac{dx'}{dt} - 2pg.\left(\frac{x'dx'}{r'dt} + \frac{y'dy'}{r'dt}\right).$$

Add these equations, observing that $r^2 = x^2 + y^2$, whence

$$\frac{dr}{dt} = \frac{xdx}{rdt} + \frac{ydy}{rdt}; \text{ similarly, } \frac{dr'}{dt} = \frac{x'dx'}{r'dt} + \frac{y'dy'}{r'dt};$$

and $r + r' =$ a constant length APC; $\therefore\ \frac{dr}{dt} + \frac{dr'}{dt} = 0$. Thus we have

$$2m.\left(\frac{dx}{dt}\,\frac{d^2x}{dt^2} + \frac{dy}{dt}\,\frac{d^2y}{dt^2}\right) + 2m'.\left(\frac{dx'}{dt}\,\frac{d^2x'}{dt^2} + \frac{dy'}{dt}\,\frac{d^2y'}{dt^2}\right).$$

$$= 2g\left(m\frac{dx}{dt} + m'\frac{dx'}{dt}\right).$$

Integrating, we have

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) + m'\left(\frac{dx'^2}{dt^2} + \frac{dy'^2}{dt^2}\right) = 2g\,(mx + m'x') + C.$$

If we suppose a, a', to be the values of x, x', when the velocities are 0, we shall have

$$m\frac{ds^2}{dt^2} + m'\frac{ds'^2}{dt^2} = m\,.\,2g\,(x - a) - m'\,.\,2g\,(a' - x')\ldots\ldots(2).$$

This equation might have been obtained at once by applying the principle of the conservation of *vis viva*, Art. 155.

By introducing into equation (2), the relations among x, x', s, s', given by the nature of the curve, and by the condition $r + r' =$ a constant quantity, we have the equations which determine the motion.

Ex. Let Q, hanging freely, draw P along a horizontal plane, fig. 24. Let the original position of Q, when the bodies begin to move, be D; $AD = a$, $AB = c$; length of string $PAQ = l$: $AP = r$, $AQ = l - r$.

$$\text{Velocity of } Q = -\frac{dr}{dt}; \text{ and } BP = \sqrt{(r^2 - c^2)};$$

$$\therefore \text{ velocity of } P = \frac{r}{\sqrt{(r^2 - c^2)}} \cdot \frac{dr}{dt};$$

P neither ascends nor descends: hence, the equation (2) gives

$$m \cdot \frac{r^2}{r^2 - c^2} \cdot \frac{dr^2}{dt^2} + m' \cdot \frac{dr^2}{dt^2} = 2m'g(l - r - a),$$

$$\frac{dr^2}{dt^2} = \frac{2m'g(l - a - r)(r^2 - c^2)}{(m + m')r^2 - m'c^2};$$

which gives the relation between r and t.

Prob. XV. *A body* P, *fig.* 25, *is fastened to two equal weights* Q, Q′, *by strings passing over pullies* A, A′, *equidistant from it, and in the same horizontal line: to determine its motion.*

The vertical line PE will bisect AA', and the body being acted upon by equal forces on the two sides of the vertical line, will not be drawn from it, and we shall only have the vertical motion to consider.

Let $AE = EA' = a$, $EP = x$; mass $P = m$, of $Q = Q' = m'$: tension of PA, or $PA' = p$; $AP = AP' = r$: hence, the accelerating force which each string exerts vertically upon P will be $\frac{pg}{m} \cdot \frac{x}{r}$; therefore

$$\left.\begin{array}{l}\dfrac{d^2x}{dt^2}=g-\dfrac{2pg}{m}\cdot\dfrac{x}{r},\ \text{from the motion of } P\\[2ex] -\dfrac{d^2r}{dt^2}=g-\dfrac{pg}{m'},\ \text{from the motion of } Q\end{array}\right\}\ldots\ldots\ldots(1).$$

From these equations, or immediately from the conservation of *vis viva*, we find

$$m\frac{dx^2}{dt^2}+2m'\frac{dr^2}{dt^2}=2mg(x-b)-4m'g(r-c)\ldots\ldots(2).$$

Supposing that when the velocities are 0, $x=b=EB$, $r=c=AB$. Hence, $c^2=a^2+b^2$.

But, since $r=\sqrt{(a^2+x^2)}$, $\dfrac{dr}{dx}=\dfrac{x}{\sqrt{(a^2+x^2)}}$; this becomes

$$\left\{m+\frac{2m'x^2}{a^2+x^2}\right\}\frac{dx^2}{dt^2}=2mg(x-b)-4m'g(r-c),$$

$$\frac{dx^2}{dt^2}=\frac{a^2+x^2}{ma^2+(m+2m')x^2}\cdot 2g\left\{m(x-b)-2m'(r-c)\right\}.$$

To find when the velocity again becomes 0, we must have

$$m(x-b)-2m'(r-c)=0;$$

or, putting for r its value $\sqrt{(a^2+x^2)}$, transposing and squaring,

$$(4m'^2-m^2)x^2-2m(2m'c-mb)x+4m'^2a^2-(2m'c-mb)^2=0,$$

the two values of x in this equation give the points where the velocity is 0; hence, b is one of them; and if b' be the other, we shall have, by the doctrine of equations,

$$b+b'=\frac{2m(2m'c-mb)}{4m'^2-m^2},$$

$$b'=\frac{4mm'c-(4m'^2+m^2)b}{4m'^2-m^2}.$$

If $2m'>m$, make $EB'=b'$, and P will go on oscillating between B and B'. If $2m'<m$, b' will be negative, and P will never come to a second point where its velocity is 0.

The velocity of P will be greatest when it is in the position in which there would be an equilibrium: for then the force is 0,

$$\text{and therefore } \frac{d^2x}{dt^2} = 0.$$

That is, when $x = \dfrac{ma}{\sqrt{(4m'^2 - m^2)}}$: see *Mechanics*, Chap. II, Prob. II.

PROB. XVI. *A weight* Q *draws a weight* P *over a fixed pully* A, *fig.* 26, P *in the mean time making small oscillations: to determine the motion.*

When the oscillations are very small, we may here, as in Prob. XI, suppose the tension to be the same as if P did not oscillate: and it will thus be found.

Let m, m' be the masses of P, Q; then $m' - m$ is the mass employed in producing motion; and $m' + m$ the mass moved: hence, the accelerating force on Q is $\dfrac{m' - m}{m' + m} g$.

But the force on Q downwards is manifestly the excess of its weight downwards, above the tension which acts upwards. Hence, if p be the tension,

$$g - \frac{pg}{m'} = \frac{m' - m}{m' + m} g; \quad \therefore p = \frac{2m'm}{m' + m}.$$

Let AM vertical $= x$, MP horizontal $= y$, $AP = r$, $AQ = l - r$;

$$\therefore \frac{d^2y}{dt^2} = -\frac{pg}{m} \cdot \frac{y}{r} = -\frac{2m'g}{m' + m} \cdot \frac{y}{r} \ldots\ldots\ldots (1).$$

CASE 1. When $m' = m$, accelerating force on $Q = 0$; $\therefore \dfrac{dr}{dt}$ is constant.

Let $\frac{dr}{dt} = -b$, P being supposed to ascend; $\therefore \frac{d^2y}{dt^2} = \frac{d^2y}{dr^2}b^2$, and equation (1) becomes

$$b^2\frac{d^2y}{dr^2} = -g \cdot \frac{y}{r}, \text{ or } \frac{d^2y}{dr^2} = -\frac{ky}{r}, \ k \text{ being constant.}$$

Let the integral of this be $y = Ar + Br^2 + Cr^3 + \&c.$

$$\left.\begin{aligned} \therefore\ \frac{d^2y}{dr^2} &= 1 \cdot 2 \cdot B + 2 \cdot 3\, Cr + \&c. \\ + \frac{ky}{r} &= kA + kBr + \&c. \end{aligned}\right\} = 0;$$

$$\text{whence } B = -\frac{kA}{2},\ C = \frac{k^2A}{2^2 . 3},\ D = -\frac{k^3A}{2^2 . 3^2 . 4},\ \&c.$$

$$\therefore\ y = A\left\{r - \frac{kr^2}{2} + \frac{k^2r^3}{2^2\,3} - \frac{k^3r^4}{2^2\,3^2\,4} + \&c.\right\}$$

By making $y = 0$, we have an equation of an infinite number of dimensions; shewing that the curve described by P cuts the axis in an infinite number of points.

In order to determine the points where y is a maximum, we must have

$$\frac{dy}{dr} = 0; \quad \text{or } 1 - kr + \frac{k^2r^2}{2^2} - \frac{k^3r^3}{2^2 3^2} + \&c. = 0,$$

(agreeing with equation (3), Prob. XII,) which gives the points of extreme deviation of the pendulum from the vertical. The times of successive oscillations will be as the differences of the successive values of r, because r diminishes uniformly*.

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* This problem has been differently and erroneously solved by some authors. No solution but an approximate one attainable. Euler, *Com. Petrop.* Tom. VIII, p. 137, &c. obtains an equation not integrable, and then observes, "Ita ut determinationem hujus motus oscillatorii, quo corpora *A* et *B* cientur, dum filum super trochleam *uniformiter* promovetur, pro casu desparato declarare simus coacti."

The equation in Case 2, is integrable for some particular values of P and Q: for instance, if $Q = 3P$.

Case 2. When m, m' are unequal.

Here $-\frac{d^2r}{dt^2}=\frac{m'-m}{m'+m}.g$; $\frac{dr}{dt}=-\frac{m'-m}{m'+m}gt$; velocity being 0 when t is 0,

$$r=a-\frac{m'-m}{m'+m}.\frac{gt^2}{2}=a-\tfrac{1}{2}ngt^2;\ \text{making}\ \frac{m'-m}{m'+m}=n;$$

$$\text{whence}\ \frac{2m'}{m'+m}=1+n.$$

And equation (1) becomes

$$\frac{d^2y}{dt^2}=-\frac{(1+n)gy}{a-\frac{1}{2}ngt^2};$$

whence y may be found by series in terms of t.

CHAP. III.

INVERSE PROBLEMS RESPECTING THE MOTION OF POINTS ON CURVES.

In the first Chapter of this Part we supposed a body to move on a curve, the curve being given, and the motion being the thing to be determined. In the present one we shall collect several questions which have occupied the attention of mathematicians, in which some property or consequence of the body's motion is given, and the curve is required.

Sect. I. CURVE OF EQUAL PRESSURE.

164. PROP. *To find the curve on which a body, descending by the force of gravity, presses equally at all points.*

Let AM, fig. 27, be the vertical abscissa $= x$, MP the horizontal ordinate $= y$; the arc of the curve s, the time t, and the radius of curvature at $P = \rho$, ρ being positive when the curve is concave to the axis; then, R being the re-action at P, we have by Art. 145, page 17,

$$R = g\frac{dy}{ds} + \frac{1}{\rho}\,\frac{ds^2}{dt^2} \ldots\ldots\ldots\ldots\ldots (1).$$

But if HM be the height due to the velocity at P, $AH = h$, we have,

$$\frac{ds^2}{dt^2} = 2g(h - x).$$

Also, if we suppose s the independent variable, we have

$$\frac{1}{\rho} = -\frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}};$$

and if the constant value of R be k, equation (1) becomes

$$k = g\frac{dy}{ds} - 2g(h-x)\frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}};$$

$$\therefore\ -\frac{k}{2g\sqrt{(h-x)}}\frac{dx}{ds} = \sqrt{(h-x)}\frac{d^2y}{ds^2} - \frac{1}{2\sqrt{(h-x)}}\frac{dy}{ds}\frac{dx}{ds}.$$

The right-hand side is obviously the differential of

$$\sqrt{(h-x)}\cdot\frac{dy}{ds};$$

hence, integrating,

$$\frac{k}{g}\cdot\sqrt{(h-x)} = \sqrt{(h-x)}\cdot\frac{dy}{ds} + C,$$

$$\frac{dy}{ds} = \frac{k}{g} - \frac{C}{\sqrt{(h-x)}}\ldots\ldots\ldots\ldots(2).$$

If $C = 0$, the curve becomes a straight line inclined to the horizon, which obviously answers the condition. The sine of inclination is $\frac{k}{g}$.

In other cases the curve is found by equation (2), putting for $\frac{dy}{ds}$, $\dfrac{\dfrac{dy}{dx}}{\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}}$, finding $\frac{dy}{dx}$, and integrating.

If we differentiate equation (2), we have

$$\frac{d^2y}{ds^2} = -\frac{C}{2(h-x)^{\frac{3}{2}}}\frac{dx}{ds};\quad \rho = -\frac{\dfrac{dx}{ds}}{\dfrac{d^2y}{ds^2}} = \frac{2(h-x)^{\frac{3}{2}}}{C}\ldots\ldots(3).$$

And if C be positive, ρ is positive, and the curve is concave to the axis.

We have the curve parallel to the axis, as at C, when $\frac{dy}{ds} = 0$, that is, when $\frac{k}{g} = \frac{C}{\sqrt{(h - x)}}$; when $x = h - \frac{C^2 g^2}{k^2}$.

When x increases beyond this, the curve approaches the axis, and $\frac{dy}{ds}$ is negative; it can never become less than -1; hence, the limit of x, as B, is found by making $\frac{k}{g} - \frac{C}{\sqrt{(h - x)}} = -1$;

$$\text{or,}\quad x = h - \frac{C^2 g^2}{(k + g)^2}.$$

If $k < g$, as the curve descends towards Z, it approximates perpetually to the inclination, the sine of which is $\frac{k}{g}$. If $k > g$, fig. 28, there will be a point when the curve becomes horizontal as at D, after which it will ascend in a form similar to the descending branch.

C is known from equation (2) or (3), if we know the pressure or the radius of curvature at a given point.

If C be negative, the curve is convex to the axis. In this case the part of the pressure arising from centrifugal force diminishes the part arising from gravity, and k must be less than g, fig. 29.

Sect. II. SYNCHRONOUS CURVES.

165. In fig. 30, let AP, AP', AP'', &c. be curves of the same kind, referred to a common base AD, and differing only in their *parameters**: a curve P, P', P'', &c. cutting them,

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* Any constant line is called a *parameter*, which occurs in the equation to a curve, and by its different values gives different magnitudes to corresponding portions of the curve. Thus the radius of a circle, and the semi-axis of a cycloid are parameters.

so that the arcs *AP*, *AP′*, *AP″*, &c. may all be described in the same time, by a body descending from *A* by gravity, is said to make them *synchronous*.

Prop. *To find the curve which cuts a given assemblage of curves, so as to make them synchronous.*

Let *AM* vertical $= x$, *MP* horizontal $= y$; y and x being connected by an equation involving a parameter a. The time down *AP* is $\int_x \frac{1}{\sqrt{(2gx)}} \frac{ds}{dx}$, the integral being taken from $x = 0$ to $x = AM$; and this must be the same for all curves whatever be a. Hence, we may put

$$\int_x \frac{1}{\sqrt{(2gx)}} \frac{ds}{dx} \dots\dots\dots (1),$$

k being a constant quantity, and in differentiating, we must suppose a variable as well as x and s. Let $\frac{ds}{dx} = p$, p being a function of x and a, which will be of 0 dimensions, because x and s are quantities of the same dimensions. Hence, $\int_x \frac{p}{\sqrt{(2gx)}} = k$, and differentiating,

$$\frac{p}{\sqrt{(2gx)}} + q \frac{da}{dx} = 0 \dots\dots\dots (2),$$

q being the differential coefficient of $\int_x \frac{p}{\sqrt{(2gx)}}$, with respect to a. Now, since p is of 0 dimensions in x and a, it is easily seen that $\int_x \frac{p}{\sqrt{(2gx)}}$ is a function whose dimensions in x and a are $\frac{1}{2}$, because the dimensions of an expression are increased by 1 in integrating. Hence, by a known property of homogenenous functions, (see *Lacroix*, Elem. Treat. Art. 266,) we have

$$\frac{p}{\sqrt{(2gx)}} \,.\, x + q \,.\, a = \tfrac{1}{2} k;$$

$$\therefore\ q = \frac{k}{2a} - \frac{p\sqrt{x}}{a\sqrt{(2g)}},$$

substituting this in equation (2), it becomes

$$\frac{p}{\sqrt{(2gx)}} + \frac{k}{2a}\frac{da}{dx} - \frac{p\sqrt{x}}{a\sqrt{(2g)}}\frac{da}{dx} = 0\ldots\ldots\ldots(3),$$

in which, if we put for a its value in x and y, we have an equation to the curve $PP'P''$.

If the given time k be the time of falling down a vertical height h, we have $k = \sqrt{\frac{2h}{g}}$, and hence, equation (3) becomes

$$p\left(a - x\frac{da}{dx}\right) + \frac{da}{dx}\sqrt{(hx)} = 0\ldots\ldots\ldots(4).$$

Ex. Let the curves AP, AP', AP'' be all cycloids of which the bases coincide with AD.

Let CD be the axis of any one of these cycloids $= 2a$, a being the radius of the generating circle. If $CN = x'$, we shall have as before

$$-\frac{ds}{dx'} = \sqrt{\frac{2a}{x'}};\ \text{and since } x' = 2a - x,$$

$$\frac{ds}{dx} = \sqrt{\frac{2a}{2a - x}}.$$

Hence, $p = \sqrt{\frac{2a}{2a-x}}$; and equation (4) becomes

$$\frac{\sqrt{(2a)}}{\sqrt{(2a-x)}}\cdot\left(a - x\frac{da}{dx}\right) + \frac{da}{dx}\sqrt{(hx)} = 0\ldots\ldots\ldots(5).$$

Let $\frac{x}{a} = u$, so that $a - x\frac{da}{dx} = a^2\frac{du}{dx}$, $x = au$;

and substituting,

$$\frac{a^2\sqrt{2}}{\sqrt{(2-u)}}\frac{du}{dx} + \frac{da}{dx}\sqrt{(hau)} = 0;$$

$$\therefore\ \frac{\sqrt{2}}{\sqrt{(2u-u^2)}}\frac{du}{da} + \frac{\sqrt{h}}{a^{\frac{3}{2}}} = 0,$$

$$\sqrt{2}\,.\,\text{arc}\,(\text{ver. sin.} = u) - \frac{2\sqrt{h}}{\sqrt{a}} = C.$$

When a is infinite, the portion AP of the cycloid becomes a vertical line, and $x = h$; $\therefore u = 0$; $\therefore C = 0$.

$$\text{Hence, } \frac{x}{a} = \text{ver. sin. } \sqrt{\frac{2h}{a}} \ldots\ldots\ldots(6).$$

From this equation a should be eliminated by the equation to the cycloid, which is

$$y = a\,.\,\text{arc.}\left(\text{ver. sin.} = \frac{x}{a}\right) - \sqrt{(2ax - x^2)} \ldots\ldots\ldots(7),$$

and we should have the equation to the curve required.

Substituting in (7) from (6), we have

$$y = \sqrt{(2ah)} - \sqrt{(2ax - x^2)},$$

$$\therefore\ \frac{dy}{dx} = \frac{\sqrt{h}}{\sqrt{(2a)}}\frac{da}{dx} - \frac{1}{\sqrt{(2ax - x^2)}}\left\{x\frac{da}{dx} + a - x\right\},$$

and eliminating $\frac{da}{dx}$ by (5),

$$\frac{dy}{dx} = -\frac{2a - x}{\sqrt{(2ax - x^2)}} = -\frac{\sqrt{(2a - x)}}{\sqrt{x}}.$$

But, differentiating (7) supposing a constant, we have in the cycloid

$$\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{(2a - x)}}.$$

And hence the curve P, P', P'', cuts all the cycloids at right angles*.

The curve $PP'P''$ will meet AD in a point B, such that the given time is that of describing the whole cycloid AB. It will meet the vertical line in a point E, such that the body falls through AE in the given time.

Cor. If instead of supposing all the cycloids to meet in the point A, we suppose them all to pass through any point C, fig. 37, their bases still being in the same line AD; a curve PP' drawn so that the times down PC, $P'C$, &c. are all equal, will cut all the cycloids at right angles. This may easily be collected from the preceding reasoning.

Sect. III. TAUTOCHRONOUS CURVES.

166. If a body move upon a curve, the curve is said to be *tautochronous*, if the time of descent *to* a given point be the same, *from* whatever point the body begin to descend. We shall consider the body as descending to the lowest point.

Prop. *When a body is acted on by a constant force in parallel lines, to find the tautochronous curve.*

Let A, fig. 31, be the lowest point, D the point from which the body falls, AB vertical, BD, MP horizontal. $AM = x$, $AP = s$, $AB = h$, the constant force $= g$.

Hence, the velocity at $P = \sqrt{\{2g(h-x)\}}$,

$$\frac{dt}{ds} = -\frac{1}{\sqrt{\{2g(h-x)\}}};$$

and the whole time of descent will be found by integrating this from $x = h$, to $x = 0$.

Now, since the time is to be the same, from whatever point D the body falls, that is, whatever be h, the integral just mentioned, taken between the limits, must be independent of h. That is, if we take the integral so as to vanish when $x = 0$, and then put h for x, h will disappear altogether from the result. This must manifestly arise from its being possible to put the result in a form involving only $\frac{x}{h}$, and functions of $\frac{x}{h}$, as $\frac{x^2}{h^2}$, &c.; that is, from its being of 0 dimensions in x and h.

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* For the subnormal of the former coincides with the subtangent of the latter, each being $\frac{y\sqrt{(2a-x)}}{\sqrt{x}}$.

Let $\frac{ds}{dx} = p$, where p depends only upon the curve, and does not involve h. Then, we have

$$t = -\int_x \frac{p}{\sqrt{\{2g(h-x)\}}}$$

$$= -\frac{1}{\sqrt{(2g)}} \int_x \left\{ \frac{p}{h^{\frac{1}{2}}} + \frac{1}{2} \cdot \frac{px}{h^{\frac{3}{2}}} + \frac{1 \,.\, 3}{2 \,.\, 4} \cdot \frac{px^2}{h^{\frac{5}{2}}} + \text{\&c.} \right\};$$

and from what has been said, it is evident, that each of the quantities

$$\int_x \frac{p}{h^{\frac{1}{2}}}, \quad \int_x \frac{px}{h^{\frac{3}{2}}}, \quad \text{and generally} \quad \int_x \frac{px^n}{h^{\frac{2n+1}{2}}},$$

must be of the form $\frac{cx^{\frac{2n+1}{2}}}{h^{\frac{2n+1}{2}}}$; that is, $\int_x px^n$ must $= cx^{\frac{2n+1}{2}}$;

$$\text{hence, } px^n = \frac{2n+1}{2} cx^{\frac{2n-1}{2}}; \quad p = \frac{2n+1}{2} \frac{c}{x^{\frac{1}{2}}};$$

$$\text{or, if } \frac{2n+1}{2} c = a^{\frac{1}{2}}, \quad p = \sqrt{\frac{a}{x}}, \quad \frac{ds}{dx} = \sqrt{\frac{a}{x}},$$

which is the property of a cycloid*.

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* Without expanding, we may reason thus. If p be a function of m dimensions in x, $\frac{p}{\sqrt{(h-x)}}$ is of $m - \frac{1}{2}$ dimensions; and as the dimensions of an expression are increased by 1 in integrating, $\int_x \frac{p}{\sqrt{(h-x)}}$ is of $m + \frac{1}{2}$ dimensions in x, and when h is put for x, of $m + \frac{1}{2}$ dimensions in h. But it ought to be independent of h, or of 0 dimensions. Hence, $m + \frac{1}{2} = 0$, $m = -\frac{1}{2}$. Therefore, $p = a^{\frac{1}{2}} x^{-\frac{1}{2}}$ as before.

167. **Prop.** *When the body is acted upon by a force tending to a centre, and varying as any function of the distance, to find the tautochronous curve.*

Let S, fig. 32, be the centre of force, A the point to which the body must descend; D the point from which it descends. Let $SA = e$, $SD = f$, $SP = r$, P being any point, $AP = s$.

Now we have velocity$^2 = C - 2\int_r P$, (Art. 143.), or, if $2\int_r P = \phi(r)$, velocity$^2 = \phi(f) - \phi(r)$; because the velocity$^2 = 0$, when $r = f$. Hence, the time of describing DA is $t = \int_s - \frac{1}{\sqrt{\{\phi(f) - \phi(r)\}}}$, taken from $r = f$, to $r = e$. And since the time must be the same whatever is D, the integral so taken must be independent of f. Let

$$\phi(r) - \phi(e) = z, \quad \phi(f) - \phi(e) = h, \quad \frac{ds}{dz} = p,$$

p depending only on the nature of the curve, and not involving f. Then

$$t = -\int_z \frac{p}{\sqrt{(h - z)}}, \text{ taken from } z = h, \text{ to } z = 0$$

$$= \int_z \frac{p}{\sqrt{(h - z)}}, \text{ from } z = 0, \text{ to } z = h.$$

And this must be independent of f, and therefore of $\phi(f)$, and of h. Hence, after taking the integral, the result must be 0, when $z = 0$; and when h is put for z, it must be independent of h. Therefore it must be of 0 dimensions in z and h. But if p be of n dimensions in z, or if $p = cz^n$, $\frac{p}{\sqrt{(h - z)}}$ will be of $n - \frac{1}{2}$ dimensions, and $\int_z \frac{p}{\sqrt{(h - z)}}$ of $n + \frac{1}{2}$.

Hence, $n + \frac{1}{2} = 0$, $n = -\frac{1}{2}$, and $p = \sqrt{\frac{c}{z}}$.

Therefore $\frac{ds}{dr} = \frac{dz}{dr}\sqrt{\frac{c}{z}} = \phi' r \sqrt{\frac{c}{\phi(r) - \phi(e)}}$;

whence the curve is known.

If v be the angle ASP, we have

$$\frac{ds^2}{dr^2} = 1 + r^2 \frac{dv^2}{dr^2}, \quad \frac{dv^2}{dr^2} = \frac{1}{r^2}\left(\frac{ds^2}{dr^2} - 1\right),$$

whence we may find a polar equation to the curve.

Ex. 1. Let the force vary as the distance, and be attractive.

Here $P = mr$, $\phi(r) = mr^2$;

$$z = \phi(r) - \phi(e) = m(r^2 - e^2); \quad \frac{dz}{dr} = 2mr,$$

$$\frac{ds}{dr} = \frac{dz}{dr}\sqrt{\frac{c}{z}} = 2mr\sqrt{\frac{c}{m(r^2 - e^2)}} = r\sqrt{\frac{4cm}{r^2 - e^2}}.$$

When $r = e$, $\frac{ds}{dr}$ is infinite, or the curve is perpendicular to SA at A.

If SY, perpendicular upon the tangent PY, be called p, we have

$$\frac{p^2}{r^2} = 1 - \frac{dr^2}{ds^2} = 1 - \frac{r^2 - e^2}{4cmr^2},$$

$$p^2 = \frac{e^2 - (1 - 4cm)r^2}{4cm}.$$

If $e = 0$, or the body descend to the centre, this gives the logarithmic spiral.

In other cases let $1 - 4cm = \frac{e^2}{a^2}$; $\therefore$ $4cm = \frac{a^2 - e^2}{a^2}$;

$$p^2 = \frac{e^2(a^2 - r^2)}{a^2 - e^2};$$ the equation to a hypocycloid, see p. 11.

If $4cm = 1$, the curve becomes a straight line, to which SA is perpendicular at A.

If $4cm > 1$, the curve wlll be concave to the centre, and will go off to infinity.

Ex. 2. Let the force vary inversely as the square of the distance.

$P = \frac{m}{r^2}$; and as before, we shall find

$$p^2 = r^2 - \frac{r^5 (r - c)}{2mce}.$$

Sect. IV. BRACHYSTOCHRONOUS CURVES.

168. The *brachystochron* is the curve, down which a body must descend from one point to another, so that the time of descent may be less than that down any other curve, under the same circumstances.

PROP. *A body being acted upon by a force in parallel lines, in its descent from one point to another; to find the brachystochron.*

Let A, B, fig. 33, be the given points, and $AOPQB$ the required curve. Since the time down $AOPQB$ is less than down any other curve, if we take another curve $AOpQB$, which coincides with the former, except for the arc OPQ, we shall have

time down AO + time down OPQ + time down QB, less than

time down AO + time down OpQ + time down QB:

and if the times down QB be the same on the two suppositions, we shall have

time down OPQ less than time down any other arc OpQ.

The times down QB will be the same in the two cases, if the velocity at Q be the same. But it has been seen, (Art. 143,) that the velocity acquired at Q is the same, whether the body descend down $AOPQ$, or $AOpQ$[*]. Hence, it appears that if the time down $AOPQB$ be a minimum, the time down any portion OPQ is also a minimum.

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[*] This is true, whenever the body descends in a non-resisting space, or when the forces are necessarily the same in the same points.

169. Suppose a polygon to be constructed, and a body to move along it, having in each side of the polygon the velocity which the body would there acquire in consequence of the forces which act upon it. Suppose no velocity to be lost in passing from one side of the polygon to the next, and suppose also the velocity in each side to be constant during the motion in that side, and to change at the angles only: on these suppositions let the number of sides of the polygon be increased and their magnitude diminished indefinitely; the polygon will have for its limit a curve, and the motion of the body in the polygon will have for its limit the motion of a body moving along the curve acted upon by the same forces.

If the curve be the brachystochron, the time in the polygon of which the curve is the limit must be less than the time in any other polygon drawn under the same conditions.

Let in fig. 33, *OP*, *PQ* be two sides of the polygon which has the brachystochronous curve for its limit. Therefore the time of describing *OP*, *PQ* will be, by last article, less than the time of describing *Op*, *pQ*, any other two lines, under the same conditions.

Let *AN* be a line of abscissas in the direction of the force, *OL*, *PM*, *QN*, perpendicular to it: also let p be in the line *MP*: let

$AM = x$, $MP = y$, $LM = X$, $MN = X'$, $Op = S$, $pQ = S'$;
v the velocity in *OP*, v' the velocity in *PQ*.

$$\text{Hence, } \frac{S}{v} + \frac{S'}{v'} = \text{minimum.}$$

Differentiate this expression supposing y to vary: and let δ be used instead of d in the differentiation, d being reserved for the differentiations which suppose x and y to vary from one point *of the curve* to another.

Since the velocity acquired by a body in falling to every point of *Mp* is the same, we have, by the differentiation just spoken of,

$$\frac{\delta v}{\delta y} = 0, \quad \frac{\delta v'}{\delta y} = 0, \quad \text{and } \frac{1}{v}\,\frac{\delta S}{\delta y} + \frac{1}{v'}\,\frac{\delta S'}{\delta y} = 0 \ldots\ldots (1).$$

Let $y - LO = Y$, $QN - y = Y'$; we then have

$$S^2 = X^2 + Y^2, \quad S'^2 = X'^2 + Y'^2; \quad Y + Y' = QN - LO,$$

and since X, X', QN, LO do not vary,

$$\frac{\delta S}{\delta y} = \frac{Y}{S}\frac{\delta Y}{\delta y}, \quad \frac{\delta S'}{\delta y} = \frac{Y'}{S'}\frac{\delta Y'}{\delta y} = -\frac{Y'}{S'}\frac{\delta Y}{\delta y}.$$

Substituting these values in (1), we have

$$\frac{Y}{Sv} - \frac{Y'}{S'v'} = 0; \quad \text{or } \frac{Y}{Sv} = \frac{Y'}{S'v'}.$$

Therefore $\frac{Y}{Sv}$ is a quantity which is the same for two successive sides of the polygon, and therefore for all the sides of the polygon.

The limit of $\frac{Y}{Sv}$ is $\frac{1}{v}\frac{dy}{ds}$; hence in the brachystochron, which is the curvilinear limit of the polygon,

$$\frac{1}{v}\frac{dy}{ds} = \text{constant} \ldots\ldots\ldots\ldots (2);$$

where $\frac{dy}{ds}$ is a differential coefficient *for the curve.*

From this property of the brachystochron, v being known in terms of x, we may determine the nature of the curve.

Cor. 1. Let the force be gravity: then $v = \sqrt{(2gx)}$;

$$\frac{1}{\sqrt{(2gx)}}\frac{dy}{ds} = \text{constant}, \quad \frac{dy}{ds}\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{a}};$$

$\sqrt{a}$ being a constant quantity;

$$\therefore \frac{dy}{ds} = \sqrt{\frac{x}{a}},$$

which is a property of the cycloid, of which the axis is parallel to x, and of which the base passes through the point from which the body falls.

Cor. 2. If the body fall from a given point to another given point, setting off with the velocity acquired down a given height: the curve of quickest descent is a cycloid, of which the base coincides with the horizontal line, from which the body acquires its velocity.

170. Prop. *If a body be acted on by gravity, the curve of its quickest descent from a given point to a given curve, cuts the latter at right angles.*

Let *A*, fig. 35, be the given point, and *BM* the given curve; *AB* the curve of quickest descent cuts *BM* at right angles.

It is manifest the curve *AB* must be a cycloid, for otherwise a cycloid might be drawn from *A* to *B*, in which the descent would be shorter. If possible, let *AQ* be the cycloid of quickest descent, the angle *AQB* being acute. Draw another cycloid *AP*, and let *PP'* be the curve which cuts *AP*, *AQ*, so as to make the arcs *AP*, *AP' synchronous*. Then, by Art. 167, *PP'* is perpendicular to *AQ*, and therefore manifestly *P'* is between *A* and *Q*, and the time down *AP* is less than the time down *AQ*; therefore, this latter is not the curve of quickest descent. Hence, if *AQ* be not perpendicular to *BM*, it is not the curve of quickest descent*.

171. Prop. *If a body be acted on by gravity, and if* AB, *fig.* 36, *be the curve of quickest descent from the curve* AL *to the point* B; AT, *the tangent of* AL *at* A, *is parallel to* BV, *a perpendicular to the curve* AB *at* B.

If *BV* be not parallel to *AT*, draw *BX* parallel to *AT*, and falling between *BV* and *A*. In the curve *AL* take a point *a* near to *A*. Let *aB* be the cycloid of quickest descent from the point *a* to the point *B*; and *Bb* being taken equal and parallel to *aA*, let *Ab* be a cycloid equal and similar to *aB*. Since *ABV* is a right angle, the curve *BP*, which cuts off *AP* synchronous to *AB*, has *BV* for a tangent (Art. 170.). Also, ultimately *Aa* coincides with *AT*, and therefore *Bb* with *BX*.

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* The cycloid which is perpendicular to *BM* may be the cycloid of *longest* descent from *A* to *BM*.

Hence, *b* is between *A* and *P*. Hence, the time down *Ab* is less than the time down *AP*, and therefore, than that down *AB*. And hence the time down *aB* (which is the same as that down *Ab*,) is less than that down *AB*. Hence, if *BV* be not parallel to *AT*, *AB* is not the line of quickest descent from *AL* to *B*.

172. Prop. *If a body acted on by gravity descend to a given point* C, *fig.* 37, *setting off from a curve* BM, *with a velocity acquired in falling from a given horizontal line* AD, *the curve of quickest descent cuts the curve* BM *at right angles.*

As before, *BC* the curve of quickest descent, will be a cycloid, by Cor. 2 to Art. 169.

If possible, let *QC* be the cycloid of quickest descent, making *CQB* an acute angle. By Cor. 2 to Art. 169, the base of this cycloid will be in the horizontal line *AD*. Let *OPC* be another cycloid, of which the base is in *AD*. And by Cor. to Art. 166, if *PP′* cut off synchronous arcs *PC*, *P′C*, *PP′* will be perpendicular to the curves *PC*, *QC*. Hence, *P′* will fall between *Q* and *C*, and the time down *PC*, being equal to that down *P′C*, will be less than that down *QC*. Hence, if *QC* be not perpendicular to *QB*, it cannot be the curve of shortest descent.

Cor. From this it appears in what manner a cycloid must be drawn, so that it may be the curve of quickest descent from one given curve to another.

If the body descend from rest, from the curve *BM*, fig. 38, to the curve *CN*, by the action of gravity, the curve of quickest descent will be a cycloid, of which the base is the horizontal line *BE*, which cuts *CN* at right angles, and which is so situated, that the tangents to *BM* at *B*, and *CN* at *C*, are parallel.

If the body descend from the curve *BM* to the curve *CN*, the velocity being that acquired in falling from the horizontal line *AD*, the curve of quickest descent will be a cycloid, of which the base is the horizontal line *AD*, and which cuts both the curves *BM*, *CN*, at right angles.

For the brachystochron, when the length of the curve is given, see Mr Woodhouse's *Isoperimetrical Problems*, p. 122.

173. Pbop. *Supposing a body to be acted on by any forces whatever, to determine the brachystochron.*

Making the same notations and suppositions as before, *AL*, *LO*, fig. 33, being any rectangular co-ordinates; as before, the time down *OPQ* is a minimum.

Let δ be used as before to indicate the differential coefficients which result from *p*'s change of place in *MP*, *d* being confined to the differential coefficients of the curve. Hence

$$\frac{\delta\left(\frac{S}{v}+\frac{S'}{v'}\right)}{\delta y}=0,$$

$$\frac{1}{v}\frac{\delta S}{\delta y}+\frac{1}{v'}\frac{\delta S'}{\delta y}-\frac{S}{v^2}\frac{\delta v}{\delta y}-\frac{S'}{v'^2}\frac{\delta v'}{\delta y}=0\ldots\ldots(3).$$

Now we have, as before,

$$\frac{\delta S}{\delta y}=\frac{Y}{S}\frac{\delta Y}{\delta y},\quad \frac{\delta S'}{\delta y}=\frac{Y'}{S'}\frac{\delta Y'}{\delta y}=-\frac{Y'}{S'}\frac{\delta Y}{\delta y}.$$

Also $\frac{\delta v}{\delta y}=0$, for v is the velocity at 0, fig. 33, and does not vary.

Hence by (3) $\left(\frac{Y}{Sv}-\frac{Y'}{S'v'}\right)\frac{\delta Y}{\delta y}-\frac{S'}{v'^2}\frac{\delta v'}{\delta y}=0\ldots\ldots(4).$

Let $\frac{dv}{dx}=p$, then, by Taylor's Theorem, $v'=v+pX$ omitting Y^2 &c. because Y is a small element;

$$\frac{\delta v'}{\delta y}=X\frac{\delta p}{\delta y};\quad \frac{1}{v'}=\frac{1}{v}-\frac{pX}{v^{2\prime}}.$$

Hence our equation (4) becomes

$$\left(\frac{Y}{S}-\frac{Y'}{S'}\right)\frac{1}{v}\frac{\delta Y}{\delta y}+\frac{pXY'}{S'v^2}\frac{\delta Y}{\delta y}-\frac{S'X}{v'^2}\frac{\delta p}{\delta y}=0.$$

Therefore $$\frac{\dfrac{Y'}{S'}-\dfrac{Y}{S}}{X}-\frac{Y'p}{S'v}+\frac{S'}{v'}\frac{\dfrac{\delta p}{\delta y}}{\dfrac{\delta Y}{\delta y}}=0.$$

Now, by Taylor's Theorem, $\dfrac{Y'}{S'}=\dfrac{Y}{S}+\dfrac{d\dfrac{Y}{S}}{dx}X+\&c.$

Hence, at the limit, $$\frac{\dfrac{Y'}{S'}-\dfrac{Y}{S}}{X}=\frac{d\dfrac{Y}{S}}{dx}=\frac{d\dfrac{dy}{ds}}{dx}.$$

Also, at the limit, $\dfrac{Y'}{S'}=\dfrac{dy}{ds}$; $v'=v$,

$$\frac{\delta Y}{\delta y}=\frac{\delta X\dfrac{Y}{X}}{\delta y}=X\frac{\delta\dfrac{Y}{X}}{\delta y}\text{(because } X \text{ is const.)}=X\frac{\delta\dfrac{dy}{dx}}{\delta y},$$

and $\dfrac{S'}{X}=\dfrac{S'}{X'}=\dfrac{ds}{dx}$.

Hence our equation becomes

$$\frac{d\dfrac{dy}{ds}}{dx}-\frac{p}{v}\frac{dy}{ds}+\frac{1}{v}\frac{ds}{dx}\frac{\delta p}{\delta\dfrac{dy}{dx}}=0\;......(5):$$

whence, p being known, we may find the nature of the curve.

174. If the forces which act on the body at O, fig. 33, be equivalent to X in the direction of x, and Y in the direction of y,* we have

$$v\frac{dv}{dx}=X+Y\frac{dy}{dx},\text{ (Art. 143);}$$

$$p=\frac{dv}{dx}=\frac{X}{v}+\frac{Y}{v}\frac{dy}{dx}$$

$$\frac{\delta p}{\delta\dfrac{dy}{dx}}=\frac{Y}{v},$$

---

* X and Y in this Article are different from X and Y in the last Article.

because. X, Y, and v, are functions of AL and LO, and therefore not affected by δ.

Substituting these values in the equation (5) to the curve, we have

$$\frac{d\frac{dy}{ds}}{dx} - \left(\frac{X}{v^2} + \frac{Y}{v^2}\frac{dy}{dx}\right)\frac{dy}{ds} + \frac{Y}{v^2}\frac{ds}{dx} = 0;$$

$$\text{or } \frac{d\frac{dy}{ds}}{dx} - \left(\frac{X}{v^2}\frac{dy}{ds} - \frac{Y}{v^2}\frac{dx}{ds}\right) = 0 \ldots\ldots (6),$$

which will give the nature of the curve.

Cor. 1. If ρ be the radius of curvature, and x the independent variable, we have

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\frac{ds}{dx}}:$$

ρ being positive, when the curve is convex to AM;

$$\text{therefore } \frac{d\frac{dy}{ds}}{dx} = \frac{1}{\rho}; \text{ and hence, by (6),}$$

$$\frac{v^2}{\rho} = X\frac{dy}{ds} - Y\frac{dx}{ds}.$$

The quantity $\frac{v^2}{\rho}$ is the centrifugal force of the body, and is therefore that part of the pressure which arises from the centrifugal force. And $X\frac{dy}{ds} - Y\frac{dx}{ds}$ is the pressure which arises from resolving the forces perpendicular to the curve. Hence it appears, that in the brachystochron for any given forces, the parts of the pressure which arise from the given forces, and from the centrifugal force, must be equal.

Cor. 2. If we suppose the force to tend to a centre S, fig. 34, which may be assumed to be in the line AM, and P to be the whole force; also $SA = a$, $SP = r$, SY, perpendicular on the tangent $PY = p$; we have

$$X\frac{dy}{ds} - Y\frac{dx}{ds} = \text{force in } PS\text{, resolved parallel to } YS = P\,.\,\frac{p}{r},$$

$$v^2 = C - 2\textstyle\int_r P, \text{ (Art. 143)};$$

$$\therefore \frac{C - 2\int_r P}{\rho} = \frac{Pp}{r}.$$

$$\text{Also } \rho = -r\frac{dr}{dp};$$

$$\therefore C - 2\textstyle\int_r P = -Pp\dfrac{dr}{dp};$$

$$\therefore \frac{2}{p}\frac{dp}{dr} = \frac{-2P}{C - 2\int_r P}, \text{ and integrating,}$$

$p^2 = C'\,\{C - 2\int_r P\}$; whence the relation of p and r is known.

If the body begin to descend from A, $C - 2\int_r P$ must $= 0$ when $r = a$.

Ex. 1. Let the force vary directly as the distance.

$P = mr$, $C - 2\int_r P = \text{velocity}^2 = m(a^2 - r^2)$, $p^2 = C'm(a^2 - r^2)$, which agrees with the equation to the hypocycloid, p. 11.

Ex. 2. Let the force vary inversely as the square of the distance.

$$P = \frac{m}{r^2},\ C - 2\textstyle\int_r P = \dfrac{2m}{r} - \dfrac{2m}{a};$$

$$p^2 = \frac{2mC'}{a}\,.\,\frac{a - r}{r} = c^2\,.\,\frac{a - r}{r}, \text{ putting } c^2 = \frac{2mC'}{a},$$

$$r^2 - p^2 = \frac{r^3 + c^2 r - c^2 a}{r},$$

$$\frac{d\theta}{dr} = \frac{p}{r\sqrt{(r^2 - p^2)}} = \frac{c\sqrt{(a - r)}}{r\sqrt{(r^3 + c^2 r - c^2 a)}} = \frac{c}{r\sqrt{\left\{\dfrac{r^3}{a - r} - c^2\right\}}}.$$

When $r = a$, $\dfrac{d\theta}{dr} = 0$: when $r^3 + c^2 r - c^2 a = 0$, $\dfrac{d\theta}{dr}$ is infinite, and the curve is perpendicular to the radius as at B. This equation has only one root.

If we have $c = \dfrac{a}{2}$, $SB = \dfrac{a}{2}$, B being an apse,

if $c = \dfrac{a}{10}$, $SB = \dfrac{a}{5}$,

if $c = \dfrac{a}{30}$, $SB = \dfrac{a}{10}$,

if $c = \dfrac{a}{n^3 + n}$, $SB = \dfrac{a}{n^2 + 1}$.

175. Prop. *When a body moves on a given surface, to determine the brachystochron.*

Let x, y, z be rectangular co-ordinates, x being vertical; and as before, let S, S' be two successive elements of the curve: and let X, Y, Z, X', Y', Z' be the corresponding elements of x, y, z; then, since the minimum property will be true of the indefinitely small portion of the curve, we have, as before, supposing v, v' the velocities,

$$\frac{S}{v} + \frac{S'}{v'} = \text{min.}$$

$$\frac{\delta\left(\dfrac{S}{v} + \dfrac{S'}{v'}\right)}{\delta y} = 0 \ldots\ldots\ldots(1).$$

The variations indicated by δ are those which arise, supposing X, X' to be equal and constant, and Y, Z, Y' and Z' to vary.

Now, $S^2 = X^2 + Y^2 + Z^2$,

$$\therefore\ S\frac{\delta S}{\delta y} = Y\frac{\delta Y}{\delta y} + Z\frac{\delta Z}{\delta y}.$$

Also $Y + Y' = \text{const.}$ $Z + Z' = \text{const.}$

$$\text{Hence, }\left.\begin{aligned}\frac{\delta S}{dy} &= \frac{Y}{S}\frac{\delta Y}{\delta y} + \frac{Z}{S}\frac{\delta Z}{\delta y}\\ \frac{\delta S'}{\delta y} &= -\frac{Y'}{S'}\frac{\delta Y}{\delta y} - \frac{Z'}{S'}\frac{\delta Z}{\delta y}\end{aligned}\right\}\ldots\ldots\ldots(2).$$

For the sake of simplicity, we will suppose the body to be acted on only by a force in the direction of x, so that v, v' will depend on x alone, and will not be affected by the variation of Y, Z. Hence, we have by (1),

$$\frac{1}{v}\frac{\delta S}{\delta y} + \frac{1}{v'}\frac{\delta S'}{\delta y} = 0;$$

which, by substituting from (2), becomes,

$$\left\{\frac{Y'}{S'v'} - \frac{Y}{Sv}\right\}\frac{\delta Y}{\delta y} + \left\{\frac{Z'}{S'v} - \frac{Z}{Sv}\right\}\frac{dZ}{dy} = 0.$$

Also we shall have, as before,

$$\frac{\delta Y}{\delta y} = X\frac{\delta\dfrac{Y}{X}}{\delta y}, \quad \frac{\delta Z}{\delta y} = X\frac{\delta\dfrac{Z}{X}}{\delta y};$$

substituting these and dividing by X^2,

$$\frac{\dfrac{Y'}{S'v'} - \dfrac{Y}{Sv}}{X}\cdot\frac{\delta\dfrac{Y}{X}}{\delta y} + \frac{\dfrac{Z'}{S'v'} - \dfrac{Z}{Sv}}{X}\cdot\frac{\delta\dfrac{Z}{X}}{\delta y} = 0;$$

and taking the limit of these,

$$\frac{d\,.\dfrac{1}{v}\dfrac{dy}{ds}}{dx}\,\frac{\delta\dfrac{dy}{dx}}{\delta y} + \frac{d\,.\dfrac{1}{v}\dfrac{dz}{ds}}{dx}\,\frac{\delta\dfrac{dz}{dx}}{\delta y} = 0\ldots\ldots\ldots(3).$$

Let the surface be defined by an equation between x, y, z; and suppose that making y constant, we have $\frac{dz}{dx} = p$, and, making x constant, $\frac{dz}{dy} = q$; p, q being functions of x, y, z.

The limit of $\frac{Z}{Y}$ is $\frac{dz}{dy}$ or q, and therefore at the limit $z = qY$; hence, $\frac{Z}{X} = q\frac{Y}{X}$; and since q is not affected by the variations which δ denotes,

$$\frac{\delta\frac{Z}{X}}{\delta y} = q\frac{\delta\frac{Y}{X}}{\delta y};$$

or, since this is true at the limit only,

$$\frac{\delta\frac{dz}{dx}}{\delta y} = q\frac{\delta\frac{dy}{dx}}{\delta y}$$

Introducing this value in (3), we have

$$\frac{d\,.\,\frac{1}{v}\frac{dy}{ds}}{dx} + q\frac{d\,.\,\frac{1}{v}\frac{dz}{ds}}{dx} = 0 \ldots\ldots\ldots (4),$$

which equation determines the curve.

176. If we suppose the body not to be acted on by any force, v will be constant, (Art. 140.) and equation (4) will become

$$\frac{d\frac{dy}{ds}}{dx} + q\frac{d\frac{dz}{ds}}{dx} = 0.$$

BOOK II.

THE MOTION OF POINTS IN A RESISTING MEDIUM.

INTRODUCTION.

177. The preceding reasonings go upon the supposition, that a body in motion, and left to itself, would move on for ever with a uniform velocity. This would be true, if the body moved in a perfect vacuum; but when the motion takes place in any fluid or *medium*, whose density is finite, there will be a retardation, arising from the *resistance* which the medium offers to the motion of the body, and acting in a direction opposite to that of the body's motion: and if the body be acted on by any extraneous forces, the effects of the resistance will be combined with those of the forces, and the curves, &c., described by the body, as calculated in the preceding Book, will be modified in a manner which we shall consider in this.

The resistance will arise, in part, from the friction and tenacity of the fluid, but principally from its inertia, that is, from the force which the body, moving through the medium, necessarily exerts in putting the fluid particles in motion. Hence, it will be in a direction opposite to the body's motion: and if, *cæteris paribus*, the velocity be increased, the resistance will also be increased; for the body will strike more particles, and with greater violence. The *law*, according to which the resistance varies with respect to the velocity, is to be deduced from experiment: in most fluids, and for a moderate velocity, it appears to be nearly, but not accurately, as the square of the velocity. In the following problems we shall suppose various laws of resistance.

The resistance is of the nature of a *pressure*, or *moving* force, and may be represented by a weight. Hence, its effect on the body, that is, the *accelerating* force (or, as it may here be called, the *retarding force*,) is to be found by dividing this resisting force by the mass of the body.

The quantity of the resistance, (considered as accelerating or retarding force,) will of course depend upon the law of resistance, and upon a constant coefficient. Thus, if the resistance vary as the square of the velocity (v), it may be supposed equal to kv^2. The quantity k will vary with the density, and other circumstances of the fluid, and also with the mass, magnitude, and form of the body.

We may also represent the absolute quantity of the resistance thus;—since it assumes different values as the velocity varies, we may suppose a velocity such, that with it, the resistance shall be equal to gravity (g). Suppose V to be this velocity, and let the resistance vary as v^n, v being any velocity: we have then

$$V^n : v^n :: g : \frac{gv^n}{V} = \text{resistance to velocity } v.$$

V is such, that if a body were moving downwards with that velocity, it would move on for ever uniformly, for the action of gravity downwards, and of the resistance upwards, would exactly counteract each other; and the motion would be the same as if the body were acted on by no force at all.

If we suppose the resistance $= kv^n$, we have $k = \frac{g}{V^n}$.

CHAP. I.

RECTILINEAR MOTION OF A POINT IN A RESISTING MEDIUM.

178. THE formulæ $\frac{ds}{dt} = v$, $\frac{dv}{dt} = f$, $v\frac{dv}{ds} = f$, (Art. 7, 8.) are applicable to all cases of this kind, (v being the velocity, f the force in the direction of the motion, and s the space described,) provided we put for f the whole force arising from the attractions, &c., and diminished by that arising from the resistance. The forces in this Chapter are supposed to act in the line of the bodies' motion.

Sect. I. NO FORCES BUT THE RESISTANCE.

As an example of this case, we may suppose a body to move on a horizontal plane perfectly smooth.

PROP. *The resistance varying as any power of the velocity, to determine the motion.*

Let the resistance $= kv^n$: hence, in this case $f = -kv^n$, where the negative sign is used, because the force tends to diminish the velocity.

First, for the space,

$$v\frac{dv}{ds} = -kv^n,$$

$$k\frac{ds}{dv} = -\frac{1}{v^{n-1}},$$

$$ks = \frac{1}{(n-2)v^{n-2}} + C.$$

And if v_1 be the velocity of the body when $s = 0$,

$$(n-2)\,ks = \frac{1}{v^{n-2}} - \frac{1}{{v_1}^{n-2}}, \text{ if } n > 2,$$

$$(2-n)\,ks = {v_1}^{2-n} - v^{2-n}, \text{ if } n < 2.$$

If $n = 2$, this integral fails, and we must return to the original equation.

Second, for the time,

$$\frac{dv}{dt} = f = -kv^n,$$

$$k\frac{dt}{dv} = -\frac{1}{v^n},$$

$$kt = \frac{1}{(n-1)\,v^{n-1}} + C;$$

and if $t = 0$, when $v = v_1$,

$$(n-1)\,kt = \frac{1}{v^{n-1}} - \frac{1}{{v_1}^{n-1}} \text{ if } n > 1;$$

$$(1-n)\,kt = {v_1}^{1-n} - v^{1-n}, \text{ if } n < 1.$$

If $n = 1$, the integral fails, and must be obtained differently.

Ex. 1. Let $n = 1$; our expression will be

$$ks = v_1 - v;$$

or, the velocity lost is as the space*.

And in this case, $k\dfrac{dt}{dv} = -\dfrac{1}{v}$;

$$\therefore\ kt = l\frac{v_1}{v};$$

therefore, if the times increase uniformly, the velocities decrease in geometrical progression†.

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* *Principia*, Book II, Prop. 1.

† Ibid. Prop. 2.

$$\text{When } v = 0,\ s = \frac{v_1}{k},\ t = \text{inf.}$$

that is, in losing its whole velocity, a body will employ an infinite term, but will describe only a finite space.

Ex. 2. Let $n = 2$. In this case,

$$k\frac{ds}{dv} = -\frac{1}{v};$$

$$\therefore\ ks = \text{l}\,\frac{v_1}{v}.$$

$$\text{Again, } kt = \frac{1}{v} - \frac{1}{v_1};$$

$$\therefore\ \frac{1}{v_1} + kt = \frac{1}{v},\ \text{ or, if } T = \frac{1}{kv_1},$$

$$k(T+t) = \frac{1}{v}.$$

Hence, if the times $(T+t)$ increase in geometrical progression, the reciprocals of the velocities increase, or the velocities decrease, in the same progression*.

$$\text{Also } \frac{v_1}{v} = \epsilon^{ks};$$

$$\therefore\ kt = \frac{1}{v_1}\left\{\epsilon^{ks} - 1\right\},$$

$$1 + kv_1 t = \epsilon^{ks},\ \text{ or } kv_1(T+t) = \epsilon^{ks}.$$

Hence, if the times $(T+t)$ increase in geometrical progression, the spaces increase in arithmetical progression, and the spaces in the successive intervals are all equal.

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* *Principia*, Prop. 5.

If $t \propto \frac{1}{v_1}$, s is constant, and $v_1 - v \propto \frac{1}{v}$. Ibid. Prop. 6.

When $v = 0$, we have s and t both infinite, or the body will describe an infinite space, and move for an infinite time before it loses all its velocity.

Ex. 3. Let $n = 4$. In this case,

$$2ks = \frac{1}{v^2} - \frac{1}{{v_1}^2},$$

$$3kt = \frac{1}{v^3} - \frac{1}{{v_1}^3}.$$

SCHOLIUM. It is manifest, that if we suppose the body to move till all its velocity be destroyed,

if $n < 1$, both the space and the time are finite,

if $n = 1$, or > 1 and < 2, the space is finite, and the time infinite,

if $n = 2$, or > 2, both the space and the time are infinite.

179. PROP. *The resistance being* $= hv + kv^2$, *to determine the motion**.

$$\text{Here } \frac{ds}{dv} = -\frac{v}{f} = -\frac{1}{h + kv};$$

$$\therefore\ s = \frac{1}{k}\,\mathrm{l}\,\frac{h + kv_1}{h + kv},$$

$$\frac{dt}{dv} = \frac{1}{f} = -\frac{1}{hv + kv^2}$$

$$= \frac{1}{h}\left\{\frac{1}{v} - \frac{k}{h + kv}\right\}$$

$$t = \frac{1}{h}\left\{\mathrm{l}\,\frac{v}{v_1} + \mathrm{l}\,\frac{h + kv_1}{h + kv}\right\}$$

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* *Principa*, Book II, Prop. 11, 12.

$$=\frac{1}{h}\,\mathrm{l}\,\frac{v\,(h+kv_1)}{v_1\,(h+kv)}.$$

When $v=0$, $s=\frac{1}{k}\,\mathrm{l}\left(1+\frac{kv_1}{h}\right)$, $t=\text{inf.}$

Sect. II. THE BODY ACTED ON BY A CONSTANT FORCE BESIDES RESISTANCE.

180. The constant force may be supposed to be gravity $(=g)$, and the body to move in a vertical line, upwards or downwards. In the former case, the force which acts upon it is the *sum* of gravity and the resistance; in the latter, it is the *difference*. Hence, the motions in ascent and descent are not similar, and cannot be obtained from the same equation, as in the case of a vacuum. They may however easily be obtained separately, as in the following examples.

PROP. *A body is acted on by gravity, and also by a resistance varying as the velocity; to determine the motion**.

For the descent, $f=g-kv$;

$$\frac{ds}{dv}=\frac{v}{f}=\frac{v}{g-kv}$$

$$=\frac{1}{k}\left\{\frac{g}{g-kv}-1\right\};$$

$\therefore\ s=\frac{g}{k^2}\,\mathrm{l}\,\frac{g-kv_1}{g-kv}-\frac{v-v_1}{k}$ supposing $g-kv$ positive,

and v_1 being the velocity when $s=0$.

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* *Principia*, Book II, Prop. 3.

It may be observed, that the constant force of gravity, urging a body to descend in a resisting medium, is not the same as the whole force of gravity g in a vacuum. It is only the *relative gravity* of the body, with respect to the fluid; that is, its weight, diminished by the weight of an equal bulk of the fluid.

It is manifest that the greatest value which v can assume, is that which makes

$$g - kv = 0, \quad \text{or } v = \frac{g}{k} = V \text{ suppose; so that } k = \frac{g}{V}.$$

This velocity V is called the *terminal* velocity; it is the limit to which the velocity perpetually approximates as the body descends, but which it never actually attains in a finite time.

If we put $\frac{g}{V}$ for k, we have

$$s = \frac{V^2}{g} \,\mathrm{l}\, \frac{V - v_1}{V - v} - \frac{V}{g}(v - v_1).$$

$$\text{Again, } \frac{dt}{dv} = \frac{1}{f} = \frac{1}{g - kv};$$

$$\therefore \; t = \frac{1}{k} \,\mathrm{l}\, \frac{g - kv_1}{g - kv}$$

$$= \frac{V}{g} \,\mathrm{l}\, \frac{V - v}{V - v}.$$

From this we should easily obtain the relation between s and t.

If we suppose $kv - g$ positive, we have

$$s = \frac{g}{k^2} \,\mathrm{l}\, \frac{kv_1 - g}{kv - g} + \frac{v_1 - v}{k} = \frac{V^2}{g} \,\mathrm{l}\, \frac{v_1 - V}{v - V} + \frac{V}{g}(v_1 - v).$$

Hence, if v be at one point greater than V, it will always continue so: this is the case when the body is projected downwards with a velocity *greater than the terminal velocity*. The velocity will, in this case, *decrease* and approach to the terminal velocity as its limit.

The *ascent* may be obtained in nearly the same manner by making $f = -(g + kv)$.

181. **Prop.** *A body is acted upon by gravity, and by a resistance varying as a square of the velocity: to determine the ascent and descent**.

For the ascent: $f = -(g + kv^2)$,

$$\frac{ds}{dv} = \frac{v}{f} = -\frac{v}{g + kv^2};$$

$$s = \frac{1}{2k}\,\mathrm{l}\,\frac{g + kv_1^2}{g + kv^2}.$$

When $v = 0$, $s = \dfrac{1}{2k}\,\mathrm{l}\left(1 + \dfrac{kv_1^2}{g}\right)$, which gives the whole height ascended.

When k is small, or the density small,

whole height $= \dfrac{1}{2k}\left\{\dfrac{kv_1^2}{g} - \dfrac{1}{2}\cdot\dfrac{k^2v_1^4}{g^2}\right\}$, neglecting terms beyond k

$$= \frac{v_1^2}{2g} - \frac{kv_1^4}{4g^2}.$$

The first term is the ascent in a vacuum, and therefore the second is the *defect* of height produced by a medium of small density. It appears from the expression that this defect is as the square of the height ascended.

Again, $\dfrac{dt}{dv} = \dfrac{1}{f} = -\dfrac{1}{g + kv^2}$,

$$t = \frac{1}{\sqrt{(kg)}}\cdot\left\{\text{arc}\left(\tan. = v_1\sqrt{\frac{k}{g}}\right) - \text{arc}\left(\tan. = v\sqrt{\frac{k}{g}}\right)\right\}.$$

When $v = 0$, time of whole ascent $= \dfrac{1}{\sqrt{(kg)}}\cdot\text{arc}\left(\tan. = v_1\sqrt{\dfrac{k}{g}}\right)$.

If v_1 be infinite, $t = \dfrac{\pi}{2\sqrt{(kg)}}$, and is therefore finite, but the height ascended is infinite.

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* *Principia*, Book II. Prop. 8, 9.

When k is small, expanding the arc by the formula,

$$\text{arc} = \tan. - \frac{\tan.^3}{3} + \&c.$$

we have time $= \dfrac{1}{\sqrt{(kg)}}\left\{v_1\sqrt{\dfrac{k}{g}} - \dfrac{v_1^3}{3}\cdot\dfrac{k^{\frac{3}{2}}}{g^{\frac{3}{2}}}\right\}$ neglecting higher powers

$$= \frac{v_1}{g} - \frac{kv_1^3}{3g^2};$$

of which expression the first term is the whole time of ascent in a vacuum, and the second the *diminution* of the time by the resistance.

For the descent: $f = g - kv^2$,

$\dfrac{ds}{dv} = \dfrac{v}{f} = \dfrac{v}{g - kv^2}$; and if $g - kv^2$ be positive,

$s = \dfrac{1}{2k}\,\mathrm{l}\,\dfrac{g - kv_1^2}{g - kv^2}$; or if $k = \dfrac{g}{V^2}$, $V = \sqrt{\dfrac{g}{k}}$,

$$s = \frac{V^2}{2g}\,\mathrm{l}\,\frac{V^2 - v_1^2}{V^2 - v^2}.$$

But if $g - kv^2$ be negative,

$$s = \frac{1}{2k}\,\mathrm{l}\,\frac{kv_1^2 - g}{kv^2 - g}$$

$$= \frac{V^2}{2g}\,\mathrm{l}\,\frac{v_1^2 - V^2}{v^2 - V^2}.$$

V is the terminal velocity as before; and the former expression is true for the case when the body is projected with a less, and the latter, when with a greater velocity than V.

$$\frac{dt}{dv} = \frac{1}{f} = \frac{1}{g - kv^2} = \frac{1}{2\sqrt{(kg)}}\cdot\left\{\frac{\sqrt{k}}{\sqrt{g} + v\sqrt{k}} + \frac{\sqrt{k}}{\sqrt{g} - v\sqrt{k}}\right\},$$

$$t = \frac{1}{2\sqrt{(kg)}}\left\{\mathrm{l}\,\frac{\sqrt{g}+v\sqrt{k}}{\sqrt{g}+v_1\sqrt{k}} + \mathrm{l}\,\frac{\sqrt{g}-v_1\sqrt{k}}{\sqrt{g}-v\sqrt{k}}\right\}, \quad (v_1 < V)$$

$$= \frac{V}{2g}\left\{\mathrm{l}\,\frac{V+v}{V+v_1} + \mathrm{l}\,\frac{V-v_1}{V-v}\right\}$$

$$= \frac{V}{2g}\,\mathrm{l}\left\{\frac{V-v_1}{V+v_1}\,\frac{V+v}{V-v}\right\}.$$

When $v_1 > V$, the expression is similarly integrated.

If the body fall from rest, $v_1 = 0$,

$$s = \frac{V^2}{2g}\,\mathrm{l}\,\frac{V^2}{V^2 - v^2}, \quad t = \frac{V}{2g}\,\mathrm{l}\,\frac{V+v}{V-v}.$$

The last equation gives

$$\therefore\ \frac{V-v}{V+v} = \epsilon^{-\frac{2gt}{V}}.$$

When t becomes considerable, the right-hand side of this equation is very small, and v very nearly $= V$. Hence, though the body never acquires the terminal velocity, after the lapse of a certain finite time, it comes very near it, and the velocity afterwards may be considered constant.

182. **Prop.** *The resistance varying partly as the velocity, and partly as the square of the velocity: to determine the ascent of a body by gravity**.

$$\text{Resistance} = hv + kv^2; \quad f = -g - hv - kv^2;$$

$$\frac{ds}{dv} = \frac{v}{f} = -\frac{v}{g + hv + kv^2}.$$

Let $v + \frac{h}{2k} = u, \quad v^2 + \frac{hv}{k} + \frac{h^2}{4k^2} = u^2,$

$$g + hv + kv^2 = g - \frac{h^2}{4k} + ku^2,$$

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* *Principia*, Book II. Prop. 13, 14.

and we shall have two different cases, as $\frac{h^2}{4k}$ is less or greater than g.

In the former case,

$$\frac{ds}{du} = -\frac{u - \frac{h}{2k}}{g - \frac{h^2}{4k} + ku^2}$$

$$= -\frac{4ku}{4kg - h^2 + 4k^2u^2} + \frac{2h}{4kg - h^2 + 4k^2u^2},$$

$$s = -\frac{1}{2k}\ \mathrm{l}\ (4kg - h^2 + 4k^2u^2)$$

$$+ \frac{h}{k}\,\text{arc.}\left\{\text{tan.} = \frac{2ku}{\sqrt{(4kg - h^2)}}\right\} + C$$

$$= -\frac{1}{2k}\ \mathrm{l}\ 4k\,(g + hv + kv^2)$$

$$+ \frac{h}{k}.\ \text{arc.}\left\{\text{tan.} = \frac{2kv + h}{\sqrt{(4kg - h^2)}}\right\} + C$$

$$= \frac{1}{2k}\ \mathrm{l}\ \frac{g + hv_1 + kv_1^2}{g + hv + kv^2}$$

$$+ \frac{h}{k}.\left\{\text{arc. tan.} = \frac{2kv + h}{\sqrt{(4kg - h^2)}} - \text{arc. tan.} = \frac{2kv_1 + h}{\sqrt{(4kg - h^2)}}\right\}.$$

In the latter case, when $\frac{h^2}{4k}$ is greater than g, we have

$$\frac{ds}{du} = -\frac{4ku}{4k^2u^2 - (h^2 - 4kg)} + \frac{2h}{4k^2u^2 - (h^2 - 4kg)},$$

and both the terms may be integrated by logarithms.

$$\text{Also } \frac{dt}{dv} = \frac{1}{f} = -\frac{1}{g + hv + kv^2},$$

which may be integrated in the same manner.

In this case the terminal velocity V is found by solving the equation $g - hV - kV^2 = 0$. One of the roots of this will be negative, and the other gives the velocity.

Sect. III. THE BODY BEING ACTED ON BY A VARIABLE FORCE.

183. In this case, to make the problem more general, we may suppose that the density of the medium, or its power of producing resistance, is different in different places. On this supposition k will be variable.

PROP. *A body is acted on by a force* P *in the line of its motion, and by a resistance which is as the square of the velocity: to determine its motion.*

Let s be the body's distance from a given point towards which the force tends.

In the descent towards the given point $f = -P + kv^2$,

$$v\frac{dv}{ds} = -P + kv^2,$$

$$\text{or, } v\frac{dv}{ds} - kv^2 = -P \ldots\ldots\ldots\ldots\ldots (1).$$

$$\text{Now } \frac{d.v^2\epsilon^{-\int_s 2k}}{ds} = 2v\frac{dv}{ds}\epsilon^{-\int_s 2k} - 2kv^2\epsilon^{-\int_s 2k};$$

hence, the first side of equation (1), multiplied by $2\epsilon^{-\int_s 2k}$, becomes integrable.

Multiplying by this factor and integrating, we have

$$v^2\epsilon^{-\int_s 2k} = -2\int_s P\epsilon^{-\int_s 2k} + C,$$

$$v^2 = 2\epsilon^{2\int_s k}\int_s P\epsilon^{-2\int_s k} + C\epsilon^{2\int_s k} \ldots\ldots (2).$$

If the integrals on the right-hand side of equation (2) can be taken, the velocity is found. And this being known, the time may be found.

Ex. Let the force be inversely as s^2, and the density inversely as s.

Here $k = \frac{h}{s}$, h being constant; $\int_s k = h \int_s \frac{1}{s} = h \;]\; s$.

$$\epsilon^{-2\int_s k} = s^{-2h}; \text{ and the force being } = \frac{m}{s^2},$$

$$\int_s P\epsilon^{-2\int_s k} = \int_s \frac{m}{s^{2h+2}} = -\frac{2m}{(2h+1)\,s^{2h+1}};$$

$$\therefore\ v^2 = Cs^{2h} - \frac{2m}{(h+1)\,s}.$$

And if $v = 0$, when $s = a$, $C = \frac{2m}{(2h+1)\,a^{2h+1}}$;

$$\therefore\ v^2 = \frac{2m}{(2h+1)\,a}\left\{\frac{s^{2h}}{a^{2h}} - \frac{a}{s}\right\}.$$

For the time, $\frac{dt}{ds} = \frac{1}{v}$; and putting for v its value, the possibility of the integration will depend on the value of h.

In the same manner, if the force vary as any power of the distance, the velocity may be found.

If the density vary inversely as the square of the distance, and the force as any inverse power of the distance, the integration is possible.

If the density vary inversely as the cube of the distance, and the force as any inverse odd power, the integration is possible.

And generally, if the density vary as any inverse power r, of the distance, the integration is possible, if the force vary as any inverse power of the distance, whose index is contained in the series, r, $2r-1$, $3r-2$, &c.

CHAP. II.

THE FREE CURVILINEAR MOTION OF A BODY IN A RESISTING MEDIUM.

184. THE second law of motion, and the results deduced from it, are true when the body moves in a resisting medium, provided we comprehend the resistance among the forces which operate to change the motion. Hence, we may apply to this case also, the equations (a) Art. 19, (the motion being in the same plane,) viz.

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y,$$

taking care to include in X, Y the resolved parts of the resistance in the directions of x, y.

Let R be the resistance in the direction of the curve, and opposite to the body's motion. Then it is clear, that the resolved parts of this in the direction of x and y are

$$-R\frac{dx}{ds}, \quad -R\frac{dy}{ds};$$

where the negative sign is used, because when x and y are increasing, the resistance tends to diminish them. Adding these to the other forces which act on the body, we have the whole values of X and Y, and may proceed to determine the motion nearly in the same manner as in the former book.

If the motion of the body be not all in the same plane, we shall have, besides the two equations already mentioned, a third $\frac{d^2z}{dt^2} = Z$; and Z will involve a term $-R.\frac{dz}{ds}$, arising from the resistance.

Sect. I. THE FORCE ACTING IN PARALLEL LINES AND CONSTANT.

185. PROP. *Let a body, acted on by a constant force, as gravity,* (= g), *be projected in a uniform medium, of which the resistance is as the velocity: it is required to find the curve**, fig. 39.

Let x, measured from the point of projection A, be horizontal, y vertical, s the curve, t the time. In this case

$$R = k \cdot \frac{ds}{dt};$$

hence, our equations become

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt}, \quad \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt} \ldots\ldots\ldots(1).$$

Integrating with respect to t,

$$\mathrm{l}\,\frac{dx}{dt} = C - kt; \quad \mathrm{l}\left(g + k\frac{dy}{dt}\right) = C' - kt,$$

and when $t = 0$, if a be the velocity of the body, and α the angle which its direction makes with the horizon, we have at that time

$$\frac{dx}{dt} = a \cos.\alpha, \quad \frac{dy}{dt} = a \sin.\alpha.$$

Hence, obtaining the values of C, C', and reducing,

$$\frac{dx}{dt} = a \cos.\alpha\,\epsilon^{-kt}, \quad g + k\cdot\frac{dy}{dt} = (g + ka \sin.\alpha)\,\epsilon^{-kt} \ldots (2).$$

Now integrate the first of these, and substitute in the second;

$$x = -\frac{a \cos.\alpha\,\epsilon^{-kt}}{k} + \text{constant}$$

$$= \frac{a \cos.\alpha\,(1 - \epsilon^{-kt})}{k}, \text{ for } x = 0, \text{ when } t = 0;$$

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* *Principia,* Book II, Prop. 4.

$$\therefore\ 1 - \epsilon^{-kt} = \frac{kx}{a\cos.\alpha};\quad \epsilon^{-kt} = 1 - \frac{kx}{a\cos.\alpha}.$$

$$-kt = \mathrm{l}\left(1 - \frac{kx}{a\cos.\alpha}\right);\quad \therefore\ \frac{dt}{dx} = \frac{1}{a\cos.\alpha - kx}.$$

Putting the values of $\frac{dt}{dx}$ and of ϵ^{-kt} in the second of equations (2),

$$g + k(a\cos.\alpha - kx)\frac{dy}{dx} = (g + ka\sin.\alpha)\left(\frac{a\cos.\alpha - kx}{a\cos.\alpha}\right);$$

$$\frac{dy}{dx} = \frac{g + ka\sin.\alpha}{ka\cos.\alpha} - \frac{g}{k}.\frac{1}{a\cos.\alpha - kx}\ \ldots\ldots(3).$$

Integrating, y being $= 0$, when $x = 0$,

$$y = \left(\tan.\alpha + \frac{g}{ka\cos.\alpha}\right)x - \frac{g}{k^2}.\ \mathrm{l}\,\frac{a\cos.\alpha}{a\cos.\alpha - kx}\ \ldots(4),$$

the equation to the path of the projectile.

Cor. 1. Expanding the logarithm, this becomes

$$y = \tan.\alpha.x + \frac{gx}{ka\cos.\alpha} - \frac{gx}{ka\cos.\alpha} - \frac{1}{2}.\frac{gx^2}{a^2\cos.^2\alpha} - \frac{1}{3}.\frac{gkx^3}{a^3\cos.^3\alpha} + \&c.$$

where, omitting the terms which destroy each other, the two first terms represent a parabola, the curve described in a vacuum; and the remaining ones the alteration introduced by resistance.

Cor. 2. It is manifest that x cannot become greater than $\frac{a\cos.\alpha}{k}$, for if it were so, $a\cos.\alpha - kx$ would be negative, and its logarithm impossible. Hence, if we take $AB = \frac{a\cos.\alpha}{k}$, a vertical line BD will be an asymptote to the descending branch.

Cor. 3. If we suppose the curve to be continued backwards beyond A, it will not have an asymptote to the branch

AO, but it will perpetually approximate to a certain angle of inclination with the horizon. To find this angle, make, in equation (3), x infinite and negative: we have thus,

$$\frac{dy}{dx} = \tan.\, a + \frac{g}{ka \cos.\, a},$$

for the ultimate position of the tangent at 0.

Cor. 4. If we draw AE parallel to this tangent, we have, since $AB = \frac{a \cos.\, a}{k}$;

$$BE = AB \cdot \frac{dy}{dx} = \frac{a \sin.\, a}{k} + \frac{g}{k^2}.$$

Also $BC = AB \cdot \tan.\, a = \frac{a \sin.\, a}{k}$. Hence, $CE = \frac{g}{k^2}$.

Also $AC = AB \sec.\, a = \frac{a}{k}$.

Hence, $AC : CE :: ka : g ::$ initial resistance : gravity*.

Cor. 5. To find when the body is highest, we must have $\frac{dy}{dx} = 0$; hence, by (3),

$$x = \frac{a^2 \sin.\, a \cos.\, a}{g + ka \sin.\, a}.$$

186. Prop. *Let a body, acted on by gravity, be projected in a uniform medium, in which the resistance is as the square of the velocity : it is required to find the curve*, fig. 40.

The co-ordinates are measured from the point of projection A, as before.

We have here $R = k \cdot \left(\frac{ds^2}{dt^2}\right)$, and our equations become

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* We shall have $PQ = CE \, 1 \frac{AB}{MB}$.

$$\left.\begin{array}{l}\dfrac{d^2x}{dt^2} = -k \,.\, \dfrac{ds}{dt}\dfrac{dx}{dt} \\[2ex] \dfrac{d^2y}{dt^2} = -g - k\dfrac{ds}{dt}\dfrac{dy}{dt}\end{array}\right\}\text{.........(1).}$$

The first gives, integrating with respect to t,

$$\frac{d^2x}{dt^2} = -k\frac{ds}{dt} \,.\, \frac{dx}{dt},$$

$$\frac{dx}{dt} = C\,\epsilon^{-ks}\text{.................(2).}$$

C being a constant quantity, which is to be so taken that when $s = 0$, $\frac{dx}{dt}$ may be the velocity of projection in a horizontal direction. Hence, if a be the velocity, and α the angle of projection BAC, we have

$$a \cos.\, \alpha = C;$$

$$\therefore\ \frac{dx}{dt} = a \cos \alpha\, \epsilon^{-ks},\quad \frac{dt}{dx} = \frac{\epsilon^{ks}}{a \cos.\, \alpha}.$$

Let at any point $\frac{dy}{dx} = p$, whence p will be the trigonometrical tangent of the angle TPN, which the curve at P, or its tangent PT, makes with the horizontal line PN. Hence, we have

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} + p\frac{d^2x}{dx^2},$$

which values, substituted in the second of our equations (1), give

$$\frac{dp}{dt}\frac{dx}{dt} + p\frac{d^2x}{dt^2} = -g - k \,.\, p\frac{ds}{dt}\frac{dx}{dt},$$

and by the first,

$$p\frac{d^2x}{dt^2} = -kp\frac{ds}{dt}\frac{dx}{dt};$$

$$\therefore \ \frac{dp}{dt}\frac{dx}{dt} = -g \ldots\ldots\ldots(3),$$

hence $\dfrac{dp}{dx}\left(\dfrac{dx}{dt}\right)^2 = -g$, and putting for $\dfrac{dx}{dt}$ its value already found;

$$\frac{dp}{dx} = -\frac{g\epsilon^{2ks}}{a^2\cos.^2\alpha}.$$

This equation expresses the nature of the curve. It may be made integrable, by multiplying it by the equation

$$\frac{dx}{ds}\sqrt{(1+p^2)} = 1, \text{ which gives}$$

$$\frac{dp}{ds}\sqrt{(1+p^2)} = -\frac{g\epsilon^{2ks}}{a^2\cos.^2\alpha} \ldots\ldots\ldots(5).$$

Integrating this equation, with respect to s, we have

$$p\sqrt{(1+p^2)} + \mathrm{l}\,\{p+\sqrt{(1+p^2)}\} = C - \frac{g\epsilon^{2ks}}{ka^2\cos.^2\alpha} \ldots\ldots(6);$$

where C is to be such, that when $s = 0$, $p = \tan.\alpha = \dfrac{\sin.\alpha}{\cos.\alpha}$.

Hence,

$$\frac{\sin.\alpha}{\cos.^2\alpha} + \mathrm{l}\,\frac{\sin.\alpha + 1}{\cos.\alpha} = C - \frac{g}{ka^2\cos.^2\alpha}.$$

We may eliminate ϵ^{2ks} between equations (4) and (6), and thus obtain $\dfrac{dx}{dp}$ in terms of p; and, by integrating, x in terms of p: we have also $\dfrac{dy}{dx} = p$, and hence, y in terms of p. Finally, we have $\left(\dfrac{dt}{dx}\right)^2 = -\dfrac{1}{g}\dfrac{dp}{dx}$, and for $\dfrac{dp}{dx}$ we may substitute its value, and thus by extracting the root, and integrating, obtain t in p. If these integrations could be performed, and p eliminated, we should have a complete solution of the problem.

Cor. 1. If we make $k = 0$ in the value of $\frac{dp}{ds}\sqrt{(1+p^2)}$, we have an equation which belongs to a projectile in a non-resisting medium; the original velocity and direction of projection being the same as in this problem.

Let AP be the curve in a resisting, Ap in a non-resisting medium; and let these be taken, so that the tangents PT, pt are always parallel; therefore the values of p are the same in the two cases; $AP = s$, $Ap = S$. Then by (5),

$$-\frac{g}{a^2\cos.^2\alpha}\,\frac{dS}{ds} = \frac{dp}{ds}\sqrt{(1+p^2)}, \text{ making } k = 0;$$

and $-\frac{g\epsilon^{2ks}}{a^2\cos.\alpha} = \frac{dp}{ds}\sqrt{(1+p^2)}$. Hence, p being the same in both,

$$\frac{dS}{ds} = \epsilon^{2ks}; \text{ and } 2kS = \epsilon^{2ks} - 1,$$

supposing S and s to begin together.

And hence,

$$2ks = \mathrm{l}\,(1 + 2kS).$$

Cor. 2. The quantity p will be positive, and x will increase, up to a point which is found by making $p = 0$. After that, p will be negative, and will become numerically greater and greater without limit, as the curve becomes more nearly vertical.

Cor. 3. The curve has a vertical asymptote BD, to the descending branch Pz.

To prove this, eliminate ϵ^{2ks} in (6) by means of (4), and we have

$$p\sqrt{(1+p^2)} + \mathrm{l}\,\{p + \sqrt{(1+p^2)}\} = C + \frac{1}{k}\frac{dp}{dx}.$$

Dividing by p^2,

$$\sqrt{\left(\frac{1}{p^2}+1\right)}+\frac{\mathrm{l}\,\{p+\sqrt{(1+p^2)}\}}{p^2}=\frac{C}{p^2}+\frac{1}{kp^2}\frac{dp}{dx}.$$

Now, when p becomes very large and negative, all the terms on the first side become very small, except $\sqrt{1}$*. Hence, this must ultimately be equal to the term on the other side; and we have, taking the $-$ value of $\sqrt{1}$, because p decreases as x increases,

$$-1=\frac{1}{kp^2}\frac{dp}{dx},\text{ and }\frac{dx}{dp}=-\frac{1}{kp^2}.$$

This, which is nearly true when p is very large, may be used in finding the value of x, in the interval between $p =$ a large quantity, and $p =$ infinity. And, as the value of x corresponding to the former quantity is not infinite, if the value in this interval be finite, the whole value of the abscissa will be finite, and the curve will have a vertical asymptote. Now by integrating we have,

$$x=\text{const.}+\frac{1}{kp};$$

the constant being known by giving p and x, known finite values. And when p becomes infinite and negative, we have $x =$ constant, which is finite, and there is therefore a vertical asymptote.

Cor. 4. The curve has an oblique asymptote *EH*, to *AO* the ascending branch continued backwards.

Put, in the equation $2ks=\mathrm{l}\,(1+2kS)$, $S=-\frac{1}{2k}$, and we shall have s infinite. Hence, if we take, in the parabola described in a non-resisting medium, an arc $Ao=\frac{1}{2k}$, and draw a tangent oe, the curve AO becomes ultimately parallel to oe.

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* $\mathrm{l}\,\{\sqrt{(1+p^2)}-p\}$ is infinitely small compared with p, when p is infinite; and *a fortiori*, this log. is infinitely small compared with p^2.

To shew that there is an asymptote parallel to this line, we must prove that AH is always finite, H being the intersection of the tangent with the axis.

$$AH = y\frac{dx}{dy} - x; \quad \frac{d\,.\,AH}{dx} = y\frac{d\,.\,\frac{dx}{dy}}{dx} = -\frac{y}{p^2}\frac{dp}{dx}.$$

Also, if the ultimate value of p be called n, we shall have ultimately, the abscissa being $-x$,

$$y = -nx, \quad s = -x\sqrt{(1+n^2)}; \text{ and by (4),}$$

$$\frac{dp}{dx} = -\frac{g\epsilon^{2ks}}{a^2 \cos.\, a};$$

now $AH = -\int_x \frac{y}{p^2}\frac{dp}{dx}$, will be finite, if the integral be finite when p becomes n, and when x and y become infinite.

That is, if $\int_x \frac{gx}{a^2 \cos.^2 a\,.\,n}\,.\,\epsilon^{-2kx}\sqrt{(1+n^2)}$

be finite when x is infinite; which it is, as will be seen by integrating.

COR. 5. Hence, it appears that the ascending and descending branches are not similar. The body rises more obliquely, and descends more vertically than it would do in a vacuum*.

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* For the determination of other circumstances in the curve, and for methods of constructing the curve approximately, see *Journal de l'Ec. Polyt.* tom. IV, p. 204. This part of Mechanics has been called *Ballistics*.

When the angle of elevation is small, we may modify our formulæ and obtain an approximation for that case.

For the path of the projectile when the resistance varies as $(\text{velocity})^{2n}$, see *John Bernoulli's* works, tom. II, p. 293. This is the problem which Keil proposed to foreign Mathematicians as a challenge, and of which he was unable to produce a solution when Bernoulli called upon him to compare it with his own.

Sect. II. ANY FORCE ACTING IN PARALLEL LINES.

187. PROP. *Let the force act parallel to the ordinate* y, *and be equal to* P; *to find the equations of motion**.

We shall here have, t being the independent variable,

$$\frac{d^2x}{dt^2} = -R\frac{dx}{ds},\quad \frac{d^2y}{dt^2} = -P - R\frac{dy}{ds}\ldots\ldots(1).$$

Hence, $$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} = -2P\frac{dy}{dt} - 2R\frac{ds}{dt};$$

or, $$\frac{d.\left(\frac{ds}{dt}\right)^2}{dt} = -2P\frac{dy}{dt} - 2R\frac{ds}{dt}\ldots\ldots(2).$$

Also, by (1),

$$\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} = -P\frac{dx}{dt};$$

or, $$\left(\frac{dx}{dt}\right)^2\frac{d.\frac{dy}{dx}}{dt} = -P\frac{dx}{dt},$$

$$\left(\frac{dx}{dt}\right)^2\frac{d\frac{dy}{dx}}{dt}\,\frac{dt}{dx} = -P.$$

We may here suppose x the independent variable, whence

$$\frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 = -P\ldots\ldots(3).$$

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* *Principia*, Book II, Prop. 10. This problem, which was erroneously solved in the first Edition of the Principia, gave rise to some of the most angry disputes in the controversy between the English and Foreign Mathematicians of that period.

Hence, $\dfrac{d\left(\dfrac{ds}{dt}\right)^2}{dt} = \dfrac{d.\left(\dfrac{ds}{dx}\right)^2\left(\dfrac{dx}{dt}\right)^2}{dt} = \dfrac{d.\left(\dfrac{ds}{dx}\right)^2\left(\dfrac{dx}{dt}\right)^2}{dx}.\dfrac{dx}{dt}$

is, by (3), $= -\dfrac{d.P\left(\dfrac{ds}{dx}\right)^2\dfrac{1}{\dfrac{d^2y}{dx^2}}}{dx}\dfrac{dx}{dt}.$

Putting this value in (2), and also $\frac{dy}{dx}\frac{dx}{dt}$ for $\frac{dy}{dt}$, and $\frac{ds}{dt}\frac{dx}{dt}$ for $\frac{ds}{dt}$, and omitting $\frac{dx}{dt}$ in all the terms, we have

$$\frac{d.P\left(\dfrac{ds}{dx}\right)^2\dfrac{1}{\dfrac{d^2y}{dx^2}}}{dx} = 2P\frac{dy}{dx} + 2R\frac{ds}{dx}\ldots(4).$$

This is an equation to the curve, if the resistance be known, and conversely.

Cor. 1. If P be constant and $= g$, since

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2, \quad \frac{d\left(\dfrac{ds}{dx}\right)^2}{dx} = 2\frac{dy}{dx}\frac{d^2y}{dx^2},$$

we have

$$2g\frac{dy}{dx} - g\frac{\left(\dfrac{ds}{dx}\right)^2\left(\dfrac{d^3y}{dx^3}\right)}{\left(\dfrac{d^2y}{dx^2}\right)^2} = 2g\frac{dy}{dx} + 2R\frac{ds}{dx};$$

$$R = -\frac{g}{2}\,\frac{\dfrac{ds}{dx}\,\dfrac{d^3y}{dx^3}}{\left(\dfrac{d^2y}{dx^2}\right)^2}\,\ldots\ldots\ldots(A).$$

COR. 2. In this case, if R be as the square of the velocity,

$$\text{let } R = Q\left(\frac{ds}{dt}\right)^2 = Q\left(\frac{ds}{dx}\right)^2\left(\frac{dx}{dt}\right)^2 = -Qg\frac{\left(\frac{ds}{dx}\right)^2}{\frac{d^2y}{dx^2}} \text{ by } (3);$$

$$\text{hence, by } (A),\ Q = \frac{\frac{d^3y}{dx^3}}{2\frac{ds}{dx}\,\frac{d^2y}{dx^2}} \dots\dots\dots (B),$$

which gives the density Q.

Ex. 1. A body moves in a semi-circle, acted on by a constant force in parallel lines, the resistance varying as the density and square of the velocity; it is required to find the variation of the resistance and density, fig. 41.

If we measure x from C the centre, and call the radius a, we have

$$y = \sqrt{(a^2 - x^2)},$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}};\quad \frac{ds}{dx} = \frac{a}{\sqrt{(a^2 - x^2)}},$$

$$\frac{d^2y}{dx^2} = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}},$$

$$\frac{d^3y}{dx^3} = -\frac{3a^2x}{(a^2 - x^2)^{\frac{5}{2}}}.$$

$$\text{Hence, by } (A),\ R = \frac{3ga^3x}{2a^4} = \frac{3x}{2a}\cdot g.$$

$$\text{Also velocity}^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{ds}{dx}\right)^2 \times \frac{-g}{\frac{d^2y}{dx^2}} = \frac{g(a^2 - x^2)^{\frac{3}{2}}}{a^2}.$$

Hence, density $= \dfrac{R}{\text{velocity}^2} = \dfrac{3ax}{2(a^2 - x^2)^{\frac{3}{2}}}$.

Hence, the density is $= 0$ at B, where $x = 0$; and is infinite at A. Between B and a the expression will be negative; and in order that the body may describe the arc aB, it must be *propelled* by the medium, which physically speaking, is absurd.

Ex. 2. Let the body, acted on by a constant and parallel force, move in a parabola of any order; it is required to find the resistance and density.

Let BN, fig. 41, be a tangent at the highest point; and let NP vary as any power of BN. Then if

$$CM = x, \quad MP = y, \quad CB = b,$$

we shall have $y = b - cx^n$; and differentiating

$$\frac{dy}{dx} = -ncx^{n-1}, \quad \frac{ds}{dx} = \sqrt{(1 + n^2c^2x^{2n-2})},$$

$$\frac{d^2y}{dx^2} = -n(n-1)cx^{n-2}, \quad \frac{d^3y}{dx^3} = -n(n-1)(n-2)cx^{n-3}.$$

Then by (A), $R = \dfrac{n(n-1)(n-2)cx^{n-3}\sqrt{(1+n^2c^2x^{2n-2})}\,g}{2n^2(n-1)^2c^2x^{2n-4}}$

$$= \frac{(n-2)\sqrt{(1 + n^2c^2x^{2n-2})}\,g}{2n(n-1)cx^{n-1}}.$$

If n be greater than 1, this value of R will manifestly become infinite, when $x = 0$, and negative, when x is negative. Also if NP do not generally vary as some power of BN, yet if the curve be symmetrical with respect to a line BC, it is manifest that ultimately, when BN is small, NP may be considered as proportional to some power of it. Hence, any symmetrical curve will require an infinite resistance at the vertex B; and hence, conversely, the curve described in a medium, the density of which is every where finite, cannot be symmetrical with respect to its ascending and descending branches.

Let us now take an example of a curve not symmetrical.

Ex. 3. Let the curve *OPQ*, fig. 42, be an hyperbola of any order, of which one of the asymptotes *CB* is vertical; it is required to find the resistance and density at any point*.

Let $AB = a$, $BC = b$, $AM = x$, $MP = y$. And suppose NP to be as $\frac{1}{CN^n}$.

Also let $MB = z$, so that $a - z = x$.

$$MN = \frac{bx}{a} = b - \frac{bz}{a};$$

$$NP \text{ which varies as } \frac{1}{CN^n} \text{ varies as } \frac{1}{MB^n}.$$

$$\text{Let } NP = \frac{c}{z^n};$$

$$\therefore\ y = b - \frac{bz}{a} - \frac{c}{z^n},$$

$$\frac{dy}{dz} = -\left(\frac{b}{a} - \frac{nc}{z^{n+1}}\right),$$

$$\frac{ds}{dz} = -\sqrt{\left\{1 + \left(\frac{b}{a} - \frac{nc}{z^{n+1}}\right)^2\right\}}$$

$$= -\frac{1}{z}\sqrt{\left\{z^2 + \left(\frac{bz}{a} + \frac{nc}{z^n}\right)^2\right\}},$$

$$\frac{d^2y}{dz^2} = -\frac{n(n+1)c}{z^{n+2}},$$

$$\frac{d^3y}{dz^3} = \frac{n(n+1)(n+2)c}{z^{n+3}}.$$

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* *Principia*, Book II, Prop. 10. Ex. 3.

Making these substitutions in the expression for the density, we have by (*B*)

$$Q = \frac{n(n+1)(n+2)z^{n+2}}{2n(n+1)z^{n+3}} \cdot \frac{z}{\sqrt{\left\{z^2 + \left(\frac{bz}{a} - \frac{nc}{z^n}\right)^2\right\}}}$$

$$= \frac{n+2}{2\sqrt{\left\{z^2 + \left(\frac{bz}{a} - \frac{nc}{z^n}\right)^2\right\}}}.$$

Now this does not vanish for any value of z, except $z = 0$.

Hence, if the density be every where finite, the curve will approach more nearly to this form than to the symmetrical curves before mentioned.

Sect. III. CENTRAL FORCES.

188. PROP. *A body moves in any resisting medium acted upon by any force tending to a centre; it is required to find the curve described.*

Let, in fig. 43, $SM = x$, $MP = y$, be rectangular co-ordinates to the place P of the body. $SP = r$; and P the force, any function of r, and tending to the point S. Also let R be the resistance which acts in the direction of the curve, and depends upon the velocity, and on the density of the medium: and let s be the curve described.

The resolved parts of the force P are $P\frac{x}{r}$ and $P\frac{y}{r}$ in directions Pm parallel to MS, and PM perpendicular to Pm. Also if the force R be resolved, the components in the same directions are $R\frac{dx}{ds}$, and $R\frac{dy}{ds}$; hence, we have (the body moving in the direction AP),

$$\frac{d^2x}{dt^2} = -\frac{Px}{r} - R\frac{dx}{ds},\quad \frac{d^2y}{dt^2} = -\frac{Py}{r} - R\frac{dy}{ds} \ldots\ldots (1);$$

$$\therefore\ 2\frac{dx}{dt}\frac{d^2x}{dt^2}+2\frac{dy}{dt}\frac{d^2y}{dt^2}$$

$$=-2P\left\{\frac{x}{r}\frac{dx}{dt}+\frac{y}{r}\frac{dy}{dt}\right\}-2R\left\{\frac{dx}{ds}\frac{dx}{dt}+\frac{dy}{ds}\frac{dy}{dt}\right\}.$$

But since $x^2+y^2=r^2$, $\frac{x}{r}\frac{dx}{dt}+\frac{y}{r}\frac{dy}{dt}=\frac{dr}{dt}$.

And since $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{ds}{dt}\right)^2$, $\frac{dx}{ds}\frac{dx}{dt}+\frac{dy}{ds}\frac{dy}{dt}=\frac{ds}{dt}$.

Hence the above equation becomes

$$\frac{d\left(\frac{ds}{dt}\right)^2}{dt}=-2P\frac{dr}{dt}-2R\frac{ds}{dt}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

Again, the original equations give us

$$y\frac{d^2x}{dt^2}-x\frac{d^2y}{dt^2}=-R\left(y\frac{dx}{ds}-x\frac{dy}{ds}\right).$$

But if we call the angle ASP, v, we have

$$y\frac{dx}{dt}-x\frac{dy}{dt}=r^2\frac{dv}{dt},\quad y\frac{d^2x}{dt^2}-x\frac{d^2y}{dt^2}=\frac{d\,.\,r^2\frac{dv}{dt}}{dt};$$

$$\therefore\ \frac{d\,.\,r^2\frac{dv}{dt}}{dt}=-Rr^2\frac{dv}{ds}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3).$$

In equations (2) and (3) we may make s the independent variable, and they then become

$$\frac{d\left(\frac{ds}{dt}\right)^2}{ds}=-2P\frac{dr}{ds}-2R$$

$$\frac{d\,.\,r^2\frac{dv}{ds}\frac{ds}{dt}}{ds}\frac{ds}{dt}=-Rr^2\frac{dv}{ds}.$$

R is a function of $\frac{ds}{dt}$ the velocity, and of r, if the density of the medium vary with r. Hence, putting for R its value, we may eliminate $\frac{ds}{dt}$ between the above equations, and obtain an equation in r and v, which by integration will give the curve.

189. Prop. *Under the same circumstances, it is required to find the relation between the radius vector* (r), *and the perpendicular upon the tangent* (p).

$$r^2 \frac{dv}{ds} = p, \text{ and equation (3) gives}$$

$$\frac{d \cdot p \frac{ds}{dt}}{dt} = - Rp;$$

$$p \frac{d^2 s}{dt^2} + \frac{dp}{dt} \frac{ds}{dt} = - Rp.$$

Also equation (2) gives

$$\frac{ds}{dt} \frac{d^2 s}{dt^2} = - P \frac{dr}{dt} - R \frac{ds}{dt}.$$

Eliminating R, we have

$$\left(\frac{ds}{dt}\right)^2 \frac{dp}{dt} = Pp \frac{dr}{dt}; \quad \left(\frac{ds}{dt}\right)^2 = Pp \frac{dr}{dp} \ldots\ldots (C).$$

Let q be one-fourth the chord of curvature through S: then

$$p \frac{dr}{dp} = 2q,$$

$$\left(\frac{ds}{dt}\right)^2 = 2Pq^*.$$

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[*] Hence, the velocity in the curve is equal to that acquired by falling down one-fourth the chord of curvature by the action of the constant force P; the same property which obtains in a vacuum.

And substituting in equation (2) of last Article,

$$\frac{d.Pq}{dt} = -P\frac{dr}{dt} - R\frac{ds}{dt}; \quad \therefore \quad \frac{d.Pq}{ds} = -P\frac{dr}{ds} - R,$$

$$R = -\frac{d.Pq}{ds} + P\frac{dr}{ds}\text{*} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (D).$$

When the relation between p and r is given, this formula enables us to find the resistance, and conversely.

Cor. 1. If $R = Q\left(\frac{ds}{dt}\right)^2 = Q.2Pq$; we shall have

$$Q = -\frac{1}{2Pq}\frac{d.Pq}{ds} + \frac{1}{2q}\frac{dr}{ds}\text{*} \ldots\ldots\ldots\ldots\ldots (E).$$

Ex. 1. To find the resistance in the circle, the centre of force being in the circumference, and the force as any power of the distance. The body moving towards the centre.

$$\text{Here } p = \frac{r^2}{a}, \quad P = \frac{m}{r^n},$$

$$\frac{1}{p}\frac{dp}{dr} = \frac{2}{r}; \quad q = \frac{p}{2}\frac{dr}{dp} = \frac{r}{4}; \quad \frac{ds}{dr} = -\frac{1}{\sqrt{\left(1 - \frac{r^2}{a^2}\right)}} = \frac{1}{\sin.\theta},$$

θ being the angle which r makes with a line through the centre.

$$Pq = \frac{m}{4r^{n-1}};$$

$$\frac{1}{Pq}\frac{d.Pq}{dr} + \frac{1}{q} = -\frac{n-1}{r} + \frac{4}{r} = -\frac{n-5}{r};$$

$$\therefore \quad Q = -\frac{n-5}{2r}\frac{dr}{ds} = \frac{(5-n)\sin.\theta}{2r}.$$

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* *Principia*, Book II, Props. 17, 18.

Also by (D), $R = \frac{m(5-n)}{4r^n}\sqrt{\left(1 - \frac{r^2}{a^2}\right)} = \frac{m(5-n)\sin.\theta}{4r^n}$.

Ex. 2. To find the resistance in the logarithmic spiral; the force varying as any power of the distance from the pole.

$$\text{Here,}\ p = ar; \quad \therefore\ q = \frac{p}{2}\frac{dr}{dp} = \frac{r}{2},$$

$$\left(\frac{ds}{dt}\right)^2 = 2Pq = Pr;$$

$$\therefore\ R = -\tfrac{1}{2}\frac{d.Pr}{ds} - P\frac{dr}{ds}; \quad \text{or since}\ \frac{ds}{dr} = -\frac{1}{\sqrt{(1-a^2)}},$$

(the negative sign indicating that the body is moving *towards* the centre;)

$$R = \tfrac{1}{2}\frac{d.Pr}{dr} + P.\sqrt{(1-a^2)}.$$

$$\text{And if}\ P = \frac{m}{r^n}; \quad Pr = \frac{m}{r^{n-1}};$$

$$R = \left\{-\frac{n-1}{2}.\frac{m}{r^n} + \frac{m}{r^n}\right\}\sqrt{(1-a^2)} = \frac{(3-n)}{2}.\frac{m}{r^n}\sqrt{(1-a^2)};$$

$$R = \frac{(3-n)}{2}.\frac{m}{r^n}\sqrt{(1-a^2)},$$

$$\text{and, if}\ R = Q.\text{velocity}^2 = Q.Pr = \frac{Qm}{r^{n-1}},$$

$$Q = \frac{(3-n)\sqrt{(1-a^2)}}{2r}.$$

Hence, the density is inversely as the distance from the centre*.

If $n > 3$, R and Q will become negative, and it will no longer be possible for a body to move in this spiral towards the centre; but if the motion be *from* the centre, we shall find in the same manner

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* *Principia*, Book II, Props. 15, 16.

$$Q = \frac{(n-3)\sqrt{(1-a^2)}}{2r}.$$

From the expression for R, it appears that (fig. 44.)

$$R : \text{force to centre} :: \frac{3-n}{2} Py : Sy.$$

If $n = 2$, this gives

$$R : \text{force to centre} :: Py : 2Sy.$$

Hence, when the force varies inversely as the square of the distance, and the density inversely as the distance, the resistance being small, a body may, as in fig. 44, approach the centre in a logarithmic spiral not much differing from a circle.

This motion requires a particular relation of the velocity and direction. If the velocity be not that which the direction requires, the path described will no longer be a logarithmic spiral. But if the resistance be small, the curve will still be a spiral, approaching the centre in successive oblong revolutions, as in fig. 45. And if the density, instead of being inversely as the distance, follow any other law, the orbit will still be of a similar form. We shall in a future Article consider the effect of resistance in altering an elliptical orbit.

190. Prop. *The resistance varying as the square of the velocity and the density and the force being given; it is required to find the polar equation to the trajectory.*

Taking the equations of Art. 187,

$$\frac{d\left(\frac{ds}{dt}\right)^2}{dt} = -2P\frac{dr}{dt} - 2R\frac{ds}{dt} \ldots\ldots\ldots (2).$$

$$\frac{d\,.\,r^2\dfrac{dv}{dt}}{dt} = -Rr^2\frac{dv}{ds}\ldots\ldots\ldots\ldots(3).$$

Let $R = Q\left(\dfrac{ds}{dt}\right)^2$; $\therefore$ $\dfrac{d\,.\,r^2\dfrac{dv}{dt}}{dt} = -Qr^2\dfrac{ds}{dt}\dfrac{dv}{dt}$

$$\frac{d\,.\,r^2\dfrac{dv}{dt}}{ds} = -Qr^2\frac{dv}{dt}.$$

Integrating, $r^2\dfrac{dv}{dt} = h\epsilon^{-\int_{,}Q}\ldots\ldots\ldots(F)$,

h being a constant quantity.

Now, $$\frac{ds}{dt} = \frac{\dfrac{dv}{dt}}{\dfrac{dv}{ds}} = \frac{h\epsilon^{-\int_{,}Q}}{r^2\dfrac{dv}{ds}};$$

$$\therefore\ \left(\frac{ds}{dt}\right)^2 = \frac{1}{r^4\left(\dfrac{dv}{ds}\right)^2}\times h^2\epsilon^{-2\int_{,}Q}$$

$$\frac{d\left(\dfrac{ds}{dt}\right)^2}{ds} = -\frac{2Q}{r^4\left(\dfrac{dv}{ds}\right)^2}\times h^2\epsilon^{-2\int_{,}Q} + h^2\epsilon^{-2\int_{,}Q}\,\frac{d\,\dfrac{1}{r^4\left(\dfrac{dv}{ds}\right)^2}}{ds}.$$

But by equation (2),

$$\frac{d\left(\dfrac{ds}{dt}\right)^2}{ds} = -2P\frac{dr}{ds} - 2R = -2P\frac{dr}{ds} - 2Q\left(\frac{ds}{dt}\right)^2$$

$$= -2P\frac{dr}{ds} - \frac{2Q}{r^4\left(\dfrac{dv}{ds}\right)^2}\times h^2\epsilon^{-2\int_{,}Q}.$$

Equating these values, we find

$$\frac{h^2\epsilon^{-2\int,Q}\,d\dfrac{1}{r^4\left(\dfrac{dv}{ds}\right)^2}}{ds} = -2P\frac{dr}{ds}\text{; whence}$$

$$\frac{d\dfrac{1}{r^4\left(\dfrac{dv}{ds}\right)^2}}{ds} + \frac{2P\epsilon^{\int,Q}}{h^2}\cdot\frac{dr}{ds} = 0.$$

Make $\dfrac{1}{r} = u$, whence $\dfrac{1}{r^2} = -\dfrac{du}{dr}$,

$$\frac{1}{r^4\left(\dfrac{dv}{ds}\right)^2} = \frac{1}{r^4}\left(\frac{ds}{dv}\right)^2 = \frac{1}{r^4}\left\{r^2 + \left(\frac{dr}{dv}\right)^2\right\}$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{dv}\right)^2 = u^2 + \left(\frac{du}{dr}\right)^2\left(\frac{dr}{dv}\right)^2 = u^2 + \left(\frac{du}{dv}\right)^2.$$

Substituting, differentiating with respect to v, and reducing, we have,

$$\frac{d^2u}{dv^2} + u - \frac{P\epsilon^{2\int,Q}}{h^2u^2} = 0 \ldots\ldots\ldots (G).$$

This differs from the equation (d), Art. 24. for motion in a vacuum, only in having the last term multiplied by $\epsilon^{2\int,Q}$.

If Q be small, the equation (G) will enable us to approximate to the effect of resistance in altering the orbit, as will appear in the following Article.

191. PROP. *A body acted upon by a central force, varying inversely as the square of the distance, moves in a medium of small density* (Q); *to determine the alteration in the elements of the ellipse described.*

$$\text{By } (F),\ r^2\frac{dv}{dt} = h\epsilon^{-\int_t Q};$$

or, if h' be the area in time 1 at the end of time t,

$$r^2\frac{dv}{dt} = h',\ h' = h\epsilon^{-\int_t Q}\ldots\ldots\ldots(1).$$

Also by (G), since $P = mu^2$,

$$\frac{d^2u}{dv^2} + u - \frac{m\epsilon^{2\int_t Q}}{h^2} = 0\ldots\ldots\ldots(2);$$

$$\text{or } \frac{d^2u}{dv^2} + u - \frac{m}{h'^2} = 0.$$

If h' were constant, we should find the integral as in Book I, Art. 29, and it would be

$$u = C\cos.v + D\sin.v + \frac{m}{h'^2}\ldots\ldots\ldots(3);$$

but since h' varies a little, u will differ a little from this form. We may suppose u to be still expressed by the preceding formula, C and D being now taken variable, and determined so as to satisfy equation (2).

Differentiating (3), we have

$$\frac{du}{dv} = -C\sin.v + D\cos.v + \cos.v\frac{dC}{dv} + \sin.v\frac{dD}{dv} + \frac{d.\frac{m}{h'^2}}{dv}.$$

But since we have yet only one condition to determine C, D, we may assume another: let therefore

$$\cos.v\frac{dC}{dv} + \sin.v\frac{dD}{dv} + \frac{d.\frac{m}{h'^2}}{dv} = 0\ldots\ldots\ldots(4).$$

Hence, the value of $\frac{du}{dv}$ will be reduced to its two first terms; and differentiating again,

$$\frac{d^2u}{dv^2} = -C\cos. v - D\sin. v - \sin. v\frac{dC}{dv} + \cos. v\frac{dD}{dv'};$$

substituting this in (2), it becomes

$$-\sin. v . \frac{dC}{dv} + \cos. v . \frac{dD}{dv} = 0 \ldots\ldots\ldots(5).$$

Hence, by (4) and (5), we have

$$\frac{dC}{dv} = -\cos. v . \frac{d.\frac{m}{h'^2}}{dv}, \quad \frac{dD}{dv} = -\sin. v . \frac{d.\frac{m}{h'^2}}{dv}.$$

Since by (1) $\frac{m}{h'^2} = \frac{m\epsilon^{2\int_s Q}}{h^2}$,

$$\frac{d\frac{m}{h'^2}}{ds} = \frac{m}{h^2} . 2Q\epsilon^{2\int_s Q}.$$

Now if Q and $\int_s Q$ be small, $\epsilon^{2\int_s Q}$ may be considered equal to unity; hence $\frac{d\frac{m}{h'^2}}{dv} = \frac{m}{h^2} . 2Q\frac{ds}{dv}$, nearly.

Hence, we have, as an approximation,

$$\frac{dC}{dv} = -\frac{m}{h^2} . 2Q\cos. v . \frac{ds}{dv}; \quad \frac{dD}{dv} = -\frac{m}{h^2} . 2Q\sin. v . \frac{ds}{dv} \ldots\ldots(6).$$

Now in equation (3), if h' is supposed to be constant, we may make $C = c\cos. a$, $D = c\sin. a$, and we shall have

$$u = \frac{m}{h^2} + c\cos. (v - a) = \frac{m}{h^2}\left\{1 + \frac{ch^2}{m}\cos. (v - a)\right\},$$

which agrees with the equation to an ellipse

$$u = \frac{a}{b^2}\left\{1 + e\cos.(v-\alpha)\right\},\ \text{if}\ \frac{m}{h^2} = \frac{a}{b^2} = \frac{1}{a(1-e^2)},\ \text{and}\ \frac{ch^2}{m} = e.$$

We may suppose a, e, α, in this equation, to vary in consequence of the variation of h to h'; and we shall find their variation in the following manner.

Make $C = c'\cos.\alpha'$, $D = c'\sin.\alpha'$, supposing c', α', variable; and we have

$$\cos.\alpha'\frac{dD}{dv} - \sin.\alpha'\frac{dC}{dv} = c'\frac{d\alpha'}{dv},$$

$$\sin.\alpha'\frac{dD}{dv} + \cos.\alpha'\frac{dC}{dv} = \frac{dc'}{dv}.$$

Hence, putting for $\frac{dC}{dv}$, $\frac{dD}{dv}$ the values from (6),

$$\left.\begin{aligned} c'\frac{d\alpha'}{dv} &= -\frac{m}{h'^2}2Q\sin.(v-\alpha')\frac{ds}{dv} \\ \frac{dc'}{dv} &= -\frac{m}{h'^2}2Q\cos.(v-\alpha')\cdot\frac{ds}{dv} \end{aligned}\right\}\text{.........}(7).$$

192. In order to find the effect in the course of a revolution, we must integrate these expressions through a whole circumference in the value of v; that is, from $v = \beta$, to $v = \beta + 2\pi$. Also the value of Q being very small, the change of α' and h' between these limits will be very small compared with that of v, and may be neglected. Hence, integrating

$$\int_v c'\frac{d\alpha'}{dv} = -\frac{2m}{h^2}\int_v Q\sin.(v-\alpha')\frac{ds}{dv},$$

$$\int_v \frac{dc'}{dv} = -\frac{2m}{h^2}\int_v Q\cos.(v-\alpha')\frac{ds}{dv}.$$

The integrations being taken between the value $v = \beta$, and $v = \beta + 2\pi$.

Let $\Delta a'$ be the change of a' in one revolution and $\Delta c'$ the change of c'; then

$$\int_v \frac{dc'}{dv} = \Delta c'. \quad \text{Also } \int_v c' \frac{da'}{dv}$$

is intermediate between $(c' + \Delta c')\,\Delta' a$ and $c' \Delta a'$, $\Delta c'$ being the greatest variation of c'; and therefore, neglecting $\Delta c' . \Delta a'$, we have

$$\left.\begin{aligned} c' \Delta a' &= -\frac{2m}{h'^2} \int_v Q \sin.\, (v - a') \frac{ds}{dv} \\ \Delta c' &= -\frac{2m}{h'^2} \int_v Q \cos.\, (v - a') \frac{ds}{dv} \end{aligned}\right\} \ldots\ldots\ldots (8).$$

Let Δe, Δa, $\Delta \frac{m}{h^2}$ be the changes which e, a, and $\frac{m}{h^2}$ undergo in one revolution. Then

$$\frac{m}{h'^2} = \frac{m}{h^2} \epsilon^{2\int_t Q} = \frac{m}{h^2} (1 + 2 \int_t Q), \text{ nearly};$$

$$\text{whence } \Delta \frac{m}{h'^2} = \frac{2m}{h'^2} \int_t Q = \frac{2m}{h'^2} \int_v Q \frac{ds}{dv} \ldots\ldots\ldots (9);$$

the integral being taken for a whole circumference of v, as before.

$$\text{Now since } c' = \frac{me}{h'^2}, \quad \Delta c' = \frac{m}{h'^2} \Delta e + e \Delta \frac{m}{h'^2}$$

neglecting products of Δ s.

$$\text{Hence } -\frac{2m}{h'^2} \int_v Q \cos.\, (v - a') \frac{ds}{dv} = \frac{m}{h'^2} \Delta e + \frac{2me}{h'^2} \int_v Q \frac{ds}{dv}$$

$$\Delta e = -2 \int_v Q \cos.\, (v - a') \frac{ds}{dv} - 2e \int_v Q \frac{ds}{dv} \ldots (10).$$

Again, $\frac{m}{h'^2} = \frac{1}{a(1-e^2)}$,

whence $\Delta\frac{m}{h'^2} = \frac{2e\Delta e}{a(1-e^2)} - \frac{\Delta a}{a^2(1-e^2)}$;

$$\Delta a = \frac{2ae\Delta e}{1-e^2} - a^2(1-e^2)\,\Delta\frac{m}{h'^2}$$

$$= a\left\{\frac{2e\Delta e}{1-e^2} - \frac{h'^2}{m}\,\Delta\frac{m}{h'^2}\right\};$$ which is, by (9) and (10)

$$= a\left\{-\frac{4e^2}{1-e^2}\int_v Q\frac{ds}{dv} - \frac{4e}{1-e^2}\int_v Q\cos.(v-a')\frac{ds}{dv}\right.$$
$$\left. - 2\int_v Q\frac{ds}{dv}\right\}$$

$$\Delta a = -2a\frac{1+2e^2}{1-e^2}\int_v Q\frac{ds}{dv} - \frac{4ae}{1-e^2}\int_v Q\cos.(v-a')\frac{ds}{dv}\ \ldots(11).$$

If we could perform the integrations in the values above given, we should have $\Delta a'$, Δa, Δe, for one revolution; and by the same process we might find these variations for any other period. In general however we cannot execute the integrations except by approximation, or as it is termed, by *quadratures*. See *Lacroix*, Elem. Tr. Art. 209.

193. If we suppose the excentricity to be very small, we may obtain an approximation to the above values.

If Q be any function of the distance, it will easily be seen, that, supposing e small, and neglecting powers of it, we have

$$Q\frac{ds}{dv} = A + eB\cos.(v-a'),$$

A, B being constant quantities. Hence, by (7),

$$c'\frac{da'}{dv} = -\frac{m}{h^2}\{2A\sin.(v-a') + eB\sin.2(v-a')\}.$$

PROP. *To find the effect of a resisting medium upon the elements of an elliptical orbit of small excentricity, in the course of one revolution.*

We have here, taking $\int_v$ for a whole circumference,

$$\int_v Q\frac{ds}{dv} = \int_v \{A + Be\cos.(v - a')\} = 2\pi A,$$

$$\int_v Q\sin.(v - a')\frac{ds}{dv} = \int_v \{A\sin.(v - a') + \frac{Be}{2}\sin.2(v - a')\}$$

$$= -A\cos.(v - a') - \frac{Be}{4}\cos.2(v - a'),$$

(between the limits) $= 0$,

$$\int_v Q\cos.(v - a')\frac{ds}{dv} = \int_v \{A\cos.(v - a') + \frac{Be}{2}[1 + \cos.2(v - a')]\}$$

$$= A\sin.(v - a') + \frac{Be}{4}\sin.2(v - a') + Be\pi,$$

(between the limits) $= \pi Be$.

Hence, by (8), (10), (11), of last article,

$$\Delta a' = 0,$$

$$\Delta e = -2\pi Be - 4\pi Ae = -2\pi(2A + B),$$

$$\Delta a = -2a\frac{1+2e^2}{1-e^2}2\pi A - \frac{4ae}{1-e^2}\pi Be = -\frac{4\pi a(1+2e^2)A}{1-e^2} - \frac{4\pi ae^2B}{1-e^2};$$

since e is small, e^2 may be neglected, and we have

$$\Delta a = -4\pi Aa.$$

Hence it appears that by the effect of the resistance of a medium of small density, the excentricity varies, and the major-axis diminishes perpetually, retaining the same position.

Also, if we consider the remaining terms in $\int_v Q\frac{ds}{dv}$, which will be of the form, $E\cos.2(v - a')$, $F\cos.3(v - a')$, &c. the expressions in (8) arising from these, will, when integrated through 2π, become $= 0$; hence, the truth of our result does not depend on supposing e small.

Ex. Let the density be constant,

$$r = \frac{a(1-e^2)}{1 + e \cos. (v-a)} = a - ae \cos. (v-a) + \&c.$$

$$\frac{ds}{dv} = \sqrt{\left\{r^2 + \left(\frac{dr}{dv}\right)^2\right\}} = a - ae \cos. (v-a) + \&c.$$

Hence, $A = Qa$, $B = -Qa$,

$$\Delta e = -2\pi a Q, \quad \Delta a = -4\pi a^2 Q.$$

If F be the central force, the square of the velocity of the body is Fa nearly, and QFa is the resistance to the body's motion $= R$;

$$\therefore \Delta e = -2\pi \frac{R}{F}; \quad \Delta a = -4\pi a \frac{R}{F}.$$

If the resistance be $\frac{1}{n}$ of the central force, the excentricity is diminished by the fraction $\frac{6}{n}$, and the axis is diminished by above $\frac{12}{n}$ of its value in one revolution.

CHAP. III.

THE CONSTRAINED MOTION OF A POINT ON A GIVEN CURVE IN A RESISTING MEDIUM.

194. THIS case might be treated in the same manner as those in the last Chapter, by considering, in addition to other forces, the re-action of the curve in the direction of its normal, and by resolving this force in the direction of the co-ordinates. It will, however, generally be more simple to consider only the forces along the curve. It has already been seen, that the alteration of velocity in the curve is entirely due to the resolved parts of the forces in the direction of the curve; and the same is true in a resisting medium, if we take into account the resistance, which is wholly in the same direction. Hence, we may apply the formulæ for rectilinear motion to the motion along the curve, as will be seen in the following examples.

PROB. I. *A body moves on an inverted cycloid, with a vertical axis, in a medium, the resistance of which is as the velocity; to determine the motion*, fig. 46.

Let s be the arc AP from the lowest point A. Then the force of gravity, resolved in the direction of the curve, is as AP; and if $AC = l$, force at $P = \frac{gs}{l}$, g being gravity; call this fs. Also, v being the velocity, let the resistance be $2kv$, when $2k$ is used rather than k, to avoid fractions, as will be seen. Then, by the formulæ, d = force, we shall have, when the body *descends*,

$$\frac{dv}{dt} = fs - 2kv;$$

and the same expression is true for the *ascent:* for s becomes negative, and the term which arises from gravity changes its sign as the force changes its direction. Now we have $v = -\frac{ds}{dt}$; for s decreases as t increases. Hence, t being the independent variable, we have

$$-\frac{d^2s}{dt^2} = fs + 2k\frac{ds}{dt};$$

$$\therefore\ \frac{d^2s}{dt^2} + 2k.\frac{ds}{dt} + fs = 0.$$

We shall obtain a particular integral of this by making $s = \epsilon^{mt}$, m being constant;

$$\therefore\ m^2 + 2km + f = 0,$$

$$m = -k \pm \sqrt{(-1)}.\sqrt{(f-k^2)};\ \text{or if } f - k^2 = h^2,$$

$$m = -k \pm h\sqrt{(-1)}.$$

Hence, particular integrals are

$$s = \epsilon^{-[k-h\sqrt{(-1)}]t},\quad s = \epsilon^{-[k+h\sqrt{(-1)}]t};$$

and the complete integral

$$s = C_1\epsilon^{-[k-h\sqrt{(-1)}]t} + C_2\epsilon^{-[k+h\sqrt{(-1)}]t}$$

$$= \epsilon^{-kt}\left\{C_1\epsilon^{ht\sqrt{(-1)}} + C_2\epsilon^{-ht\sqrt{(-1)}}\right\}$$

$$= \epsilon^{-kt}\left\{C_1[\cos.ht + \sqrt{(-1)}.\sin.ht] + C_2[\cos.ht - \sqrt{(-1)}\sin.ht]\right\}.$$

And if $C_1 + C_2 = C$, $(C_1 - C_2)\sqrt{(-1)} = C'$,

$$s = \epsilon^{-kt}(C\cos.ht + C'\sin.ht).$$

C and C' must be determined by considering, that at the beginning of the motion, when we shall suppose $t = 0$, we have $s = a =$ the arc AD; and the velocity $\frac{ds}{dt} = 0$.

$$\text{Now } \frac{ds}{dt} = \epsilon^{-kt}\left\{(C'h - Ck)\cos.ht - (Ch + C'k)\sin.ht\right\}.$$

Hence, when $t = 0$,

$$a = C,$$

$$0 = C'h - Ck;\ \therefore\ C' = \frac{kC}{h} = \frac{ka}{h};$$

$$\therefore\ s = \frac{a\epsilon^{-kt}}{h} . (h \cos. ht + k \sin. ht),$$

$$\frac{ds}{dt} = -\frac{a(h^2 + k^2)}{h} \epsilon^{-kt} \sin. ht = -\frac{af}{h} . \epsilon^{-kt} \sin. ht.$$

If the body descend, and then ascend till its whole velocity is destroyed, it will have performed one oscillation. This will be the case when $\frac{ds}{dt} = 0$, or $\sin. ht = 0$, which will first take place when $ht = \pi$. Hence, if T be the time of an oscillation,

$$T = \frac{\pi}{h} = \frac{\pi}{\sqrt{(f - k^2)}} = \frac{\pi}{\sqrt{\left(\frac{g}{l} - k^2\right)}}.$$

This is independent of a, and hence the oscillations occupy the same time whatever be the length of the arc. The *cycloid*, which is the *tautochronous* curve *in vacuo*, also possesses that property in a medium whose resistance is proportional to the velocity*.

Cor. 1. *To find where the velocity is the greatest.* This will be the case when the accelerating force is 0; that is, when $fs + 2k . \frac{ds}{dt} = 0$.

$$\text{Or, } f(h \cos. ht + k \sin. ht) - 2kf \sin. ht = 0;$$

$$\therefore\ \tan. ht = \frac{h}{k}.$$

Cor. 2. *To find the decrement of the arc*, that is, the difference of the arcs of descent and ascent. At the extremity of the arc of ascent we shall have $\frac{ds}{dt} = 0$, and $\therefore\ ht = \pi$. If we suppose the arc of ascent to be numerically represented by b, we shall have, putting $-b$ for s, and $\frac{\pi}{h}$ for t,

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* *Principia*, Book II. Prop. 26.

$$b = a\epsilon^{-\frac{k\pi}{h}};$$

$$\therefore\ (a - b) = a\left\{1 - \epsilon^{-\frac{k\pi}{h}}\right\},$$

and if k be small, neglecting its powers,

$$a - b = a \, . \, \frac{k\pi}{h} = \frac{\pi k a}{\sqrt{\left(\frac{g}{l} - k^2\right)}}.$$

Cor. 3. The body, after reaching the extremity of the arc of ascent, will again descend and ascend on the first side. Let the arc through which it ascends be a_1; we have then

$$a_1 = b\epsilon^{-\frac{k\pi}{h}} = a\epsilon^{-\frac{2k\pi}{h}}.$$

If the body make $2(n + 1)$ oscillations, and return to the distance a_n on the first side, we shall have

$$a_n = a_{n-1} \cdot \epsilon^{-\frac{2k\pi}{h}} = a_{n-2}\epsilon^{-\frac{4k\pi}{h}} = a\epsilon^{-\frac{2nk\pi}{h}}.$$

Hence, if a and a_n be observed, we may find $\frac{k}{h}$, and thence k.

Prob. II. *A body moves in a cycloid as before, the resistance being as the square of the velocity; to determine the motion**.

Let the force of gravity in the curve $= \frac{gs}{l} = fs$; resistance $= kv^2$; and in the descent, by the formulæ $-v\frac{dv}{ds} =$ force,

$$v\frac{dv}{ds} = -(fs - kv^2),$$

which is also true in the ascent, as in last Problem.

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* *Principia*, Book II. Prop. 29.

$$\text{Let } v^2 = 2z; \quad \therefore\ v\frac{dv}{ds} = \frac{dz}{ds};$$

$$\frac{dz}{ds} = -(fs - 2kz);$$

$$\frac{dz}{ds} - z.2k = -fs. \quad \text{Multiply by } \epsilon^{-2ks};$$

$$\therefore\ \frac{dz}{ds}\epsilon^{-2ks} - z\epsilon^{-2ks}.2k = -f\epsilon^{-2ks}s.$$

Integrating both sides,

$$z\epsilon^{-2ks} = C - f\int_s \epsilon^{-2ks}s$$

$$= C + \frac{fs\epsilon^{-2ks}}{2k} - \frac{f}{2k}\int_s \epsilon^{-2k}$$

$$= C + \frac{fs\epsilon^{-2ks}}{2k} + \frac{f\epsilon^{-2ks}}{4k^2},$$

$$z = C\epsilon^{2ks} + \frac{f}{4k^2}(2ks + 1).$$

Now at the beginning of the motion, let $s = a$, and $v = 0$, and $\therefore\ z = 0$;

$$\therefore\ 0 = C\epsilon^{2ka} + \frac{f}{4k^2}(2ka + 1);$$

$$\therefore\ C = -\frac{f\epsilon^{-2ka}}{4k^2}\{2ka + 1\},$$

$$z = \frac{f}{4k^2}\left\{2ks + 1 - (2ka + 1)\epsilon^{-2k(a-s)}\right\}.$$

Let the body perform a whole oscillation, and let b be the numerical value of the arc of ascent. Therefore putting $-b$ for s, and 0 for v, and therefore 0 for z;

$$0 = -2kb + 1 - (2ka + 1)\epsilon^{-2k(a+b)},$$

$$\text{whence } (1 - 2kb)\epsilon^{2kb} = (1 + 2ka)\epsilon^{-2ka};$$

from which b may be found from a.

Cor. 1. *To find the decrement of the arc of ascent.*

The equation last found may be put in this form,

$$\frac{1-2kb}{1+2ka} = \epsilon^{-2kb(a+b)}; \text{ and expanding,}$$

$$\left.\begin{array}{l} 1-2ka+4k^2a^2-8k^3a^3+\&\text{c.} \\ \quad -2kb+4k^2ab-8k^3a^2b+\&\text{c.} \end{array}\right\} =$$

$$1-2k(a+b)+\frac{4k^2(a+b)^2}{1\,.\,2}-\frac{8k^3(a+b)^3}{1\,.\,2\,.\,3}+\&\text{c.}$$

which gives, omitting the identical terms, and dividing by $4k^2$,

$$a(a+b)-2ka^2(a+b)+\&\text{c.} = \frac{(a+b)^2}{1\,.\,2}-\frac{2k(a+b)^3}{1\,.\,2\,.\,3}+\&\text{c.}$$

Dividing by $(a+b)$, &c. we have, omitting k^2, &c.

$$b = a - \frac{2k}{3}(5a^2-2ab-b^2).$$

Hence, the decrement of the arc $= \dfrac{2k}{3}(5a^2-2ab-b^2)$; or,

$$(\text{since } b = a \text{ nearly,}) = \frac{4ka^2}{3}, \text{ nearly.}$$

Cor. 2. *To find where the velocity is the greatest.*

We must have $\dfrac{dv}{ds} = 0$, and therefore $\dfrac{dz}{ds} = 0$;

$$\therefore\ 2k-2k(2ka+1)\,\epsilon^{-2k(a-s)} = 0;$$

$$\therefore\ \epsilon^{2k(a-s)} = 1+2ka,\ 2ka-2ks = \mathrm{l}(1+2ka).$$

If we expand, and neglect powers of k, we shall find

$$s = ka^2.$$

Prob. III. *A body oscillates in a circle, the resistance being as the square of the velocity; to find the velocity at any point.*

If, in fig. 47, the radius $CP = l$, arc $AP = s$, $ACP = \theta$, we shall have the force of gravity along the curve $= g \sin.\theta$, and the resistance

$$= k \cdot \frac{ds^2}{dt^2}.$$

Hence, since $\frac{d^2 s}{dt^2}$ = force in direction of the curve, we have

$$\frac{d^2 s}{dt^2} = -g \sin.\theta + k \frac{ds^2}{dt^2}.$$

Or because $s = l\theta$,

$$l \frac{d^2\theta}{dt^2} = -g \sin.\theta + kl^2 \cdot \frac{d\theta^2}{dt^2}.$$

Which is true both for ascent and descent.

This may be thus integrated. Multiply by $\frac{2}{l}$ and integrate, and we have

$$\frac{d\theta^2}{dt^2} = \frac{2g \cos.\theta}{l} + 2kl \int_\theta \frac{d\theta^2}{dt^2}.$$

Let $\int_\theta \frac{d\theta^2}{dt^2} = x$; $\therefore \frac{d\theta^2}{dt^2} = \frac{dx}{d\theta}$,

$$\frac{dx}{d\theta} = \frac{2g \cos.\theta}{l} + 2klx;$$

which is a linear equation of the first order, and may be integrated. Transpose, and multiply by $\epsilon^{-2kl\theta}$, and we have

$$\frac{dx}{d\theta}\epsilon^{-2kl\theta} - x\epsilon^{-2kl\theta} 2kl = \frac{2g}{l} \cdot \epsilon^{-2kl\theta} \cos\theta.$$

Where both sides are integrable. Therefore

$$x\epsilon^{-2kl\theta} = \frac{2g}{l} \int_\theta \epsilon^{-2kl\theta} \cos.\theta + C.$$

Now $\int_\theta \epsilon^{-m\theta} \cos.\theta = -\dfrac{\epsilon^{-m\theta}\cos.\theta}{m} - \dfrac{1}{m}\int_\theta \epsilon^{-m\theta}.\sin.\theta$

$$= -\frac{\epsilon^{-m\theta}\cos.\theta}{m} - \frac{1}{m}\left\{-\frac{\epsilon^{-m\theta}\sin.\theta}{m} + \frac{1}{m}\int_\theta \epsilon^{-m\theta}\cos.\theta\right\};$$

$$\therefore \left(1+\frac{1}{m^2}\right)\int_\theta \epsilon^{-m\theta}\cos.\theta = \frac{\epsilon^{-m\theta}}{m^2}\left\{\sin.\theta - m\cos.\theta\right\};$$

$$\therefore \int_\theta \epsilon^{-2kl\theta}\cos.\theta = \frac{\epsilon^{-2kl\theta}}{1+4k^2l^2}\left\{\sin.\theta - 2kl\cos.\theta\right\}.$$

And hence our equation becomes

$$z = \frac{2g}{l}.\frac{\sin.\theta - 2kl\cos.\theta}{1+4k^2l^2} + C\epsilon^{2kl\theta}.$$

Hence, $\dfrac{dz}{d\theta} = \dfrac{2g}{l}\,\dfrac{\cos.\theta + 2kl\sin.\theta}{1+4k^2l^2} + 2klC\epsilon^{2kl\theta} = \dfrac{d\theta^2}{dt^2}.$

And hence, the angular velocity is known.

If a be the value of θ at the beginning of the motion, we have

$$\frac{2g}{l}.\frac{\cos.a + 2kl\sin.a}{1+4k^2l^2} + 2klC\epsilon^{2kla} = 0;$$

$$\therefore \frac{d\theta^2}{dt^2} = \frac{2g}{l}\left\{\frac{\cos.\theta + 2kl\sin.\theta - (\cos.a + 2kl\sin.a)\,\epsilon^{-2kl(a-\theta)}}{1+4k^2l^2}\right\}.$$

Cor. 1. If we make $2kl = \tan.\beta$, we shall have

$$\frac{d\theta^2}{dt^2} = \frac{2g\cos.\beta}{l}.\left\{\cos.(\theta-\beta) - \cos.(a-\beta)\,\epsilon^{-(a-\theta)\tan.\beta}\right\}.$$

Here $\tan.\beta = \dfrac{k.2gl}{g} = \dfrac{\text{resistance to velocity acquired down } l}{\text{gravity}}.$

Cor. 2. If we make $a = \dfrac{\pi}{2} + \beta$, this becomes

$$\frac{d\theta^2}{dt^2} = \frac{2g\cos.\beta}{l}\cos.(\theta-\beta).$$

Hence, we have this curious property. If in fig. 48, we take AT horizontal, such that $AC : AT ::$ gravity : resistance to velocity acquired down AC, and make DE a quadrant, the body, setting off from D, will move as if it were acted on only by a uniform force $g \cos. \beta$ in a direction parallel to AT.

In this case, the angular velocity is greatest at E, when $\theta = \beta$, and vanishes again when $\theta - \beta = -\frac{\pi}{2}$, or $\theta = -\left(\frac{\pi}{2} - \beta\right)$.

The whole arc described DAd is a semi-circle, and the decrement of the arc of ascent is 2β.

Cor. 3. Generally, to find when the angular velocity is greatest; the expression for $\frac{d\theta^2}{dt^2}$ must be a maximum. Therefore

$$-\sin. (\theta - \beta) - \cos. (\alpha - \beta) \tan. \beta \epsilon^{-(\alpha - \theta) \tan. \beta} = 0.$$

And if the resistance, and therefore $\tan. \beta$, be very small,

$$\sin. (\theta - \beta) = -\cos. (\alpha - \beta) \tan. \beta.$$

Or, putting $\theta - \beta$, and β, for $\sin. (\theta - \beta)$, and $\tan. \beta$,

$$\theta = \beta \{1 - \cos. (\alpha - \beta)\} = 2\beta . \sin.^2 \frac{\alpha - \beta}{2}$$

$$= 2\beta . \sin.^2 \frac{\alpha}{2}, \text{ nearly.}$$

Prob. IV. *To find the time of oscillation in a cycloidal arc, in a medium, the resistance of which is as the square of the velocity.*

We have, (see Prob. II,)

$$v^2 = 2z = \frac{f}{2k^2} \{2ks + 1 - (2ka + 1) \epsilon^{-2k(a-s)}\}.$$

Put n for $2k$, and y for $a - s$, whence, $s = a - y$, and

$$v^2 = \frac{2f}{n^2}\left\{1 + ns - (1 + na)\,\epsilon^{-ny}\right\}.$$

Expanding the exponential, we have

$$v^2 = \frac{2f}{n^2}\left\{\frac{n^2}{1\,.\,2}(2ay - y^2) - \frac{n^3}{1\,.\,2\,.\,3}(3ay^2 - y^3)\right.$$
$$\left. + \frac{n^4}{1\,.\,2\,.\,3\,.\,4}(4ay^3 - y^4) - \&\text{c.}\right\}$$

$$= f\left\{2ay - y^2 - \frac{n}{3}(3ay^2 - y^3) + \frac{n^2}{3\,.\,4}(4ay^3 - y^4) - \&\text{c.}\right\}$$

$$= f\left\{A - Bn + Cn^2 - \&\text{c.}\right\} \text{ suppose;}$$

$$\therefore\ \frac{1}{v} = \frac{1}{\sqrt{f}}\left\{A - Bn + Cn^2 - \&\text{c.}\right\}^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{f}}\left\{\frac{1}{A^{\frac{1}{2}}} + n\,.\,\frac{B}{2A^{\frac{3}{2}}} + n^2\left(\frac{1\,.\,3\,.\,B^2}{2\,.\,4\,.\,A^{\frac{5}{2}}} - \frac{C}{2A^{\frac{3}{2}}}\right) + \&\text{c.}\right\}.$$

And $\dfrac{dt}{dy} = -\dfrac{dt}{ds}\dfrac{ds}{dy} = \dfrac{1}{v}$; whence, t will be found by integrating each term of the above series with respect to y.

To find the time of descending to the lowest point, (down DA, fig. 46), we must take the integrals from $y = 0$, to $y = a$.

Now $\displaystyle\int_y \frac{1}{A^{\frac{1}{2}}} = \int_y \frac{1}{\sqrt{(2ay - y^2)}} = \text{arc}\left(\text{ver. sin.} = \frac{y}{a}\right)$.

And to the lowest point $= \dfrac{\pi}{2}$.

Also $\displaystyle\int_y \frac{B}{2A^{\frac{3}{2}}} = \int_y \frac{3ay^2 - y^3}{6(2ay - y^2)^{\frac{3}{2}}} = \frac{y^2}{6(2ay - y^2)^{\frac{1}{2}}}$,

and to the lowest point $= \dfrac{a}{6}$.

Again, $\int_y \left\{\frac{1 \cdot 3 B^2}{2 \cdot 4 \cdot A^{\frac{5}{2}}} - \frac{C}{2A^{\frac{3}{2}}}\right\} = \int_y \frac{3B^2 - 4AC}{8A^{\frac{5}{2}}}$

$$= \int_y \frac{a^2 y^4}{24\,(2ay - y^2)^{\frac{5}{2}}} = \frac{a^2}{18}\,\frac{2y^3 - 3ay^2}{(2ay - y^2)^{\frac{3}{2}}}$$

$$+ \frac{a^2}{24} \text{arc} \left(\text{ver. sin.} = \frac{y}{a}\right),$$

and, to the lowest point $= \frac{a^2}{24} \cdot \frac{\pi}{2} - \frac{a^2}{18}$.

Hence, omitting terms involving n^3, &c., we have

$$\text{time down } DA = \frac{1}{\sqrt{f}}\left\{\frac{\pi}{2} + \frac{na}{6} + \frac{n^2 a^2}{24}\left(\frac{\pi}{2} - \frac{4}{3}\right)\right\}.$$

The expression for the time up AD would be the same as the above, except that the resistance acts in the opposite direction, and consequently it will be had by making n negative. Hence, if AE be b, we shall have

$$\text{time up } AE = \frac{1}{\sqrt{f}}\left\{\frac{\pi}{2} - \frac{nb}{6} + \frac{n^2 b^2}{24}\left(\frac{\pi}{2} - \frac{4}{3}\right)\right\}:$$

and the time of an oscillation through DAE

$$= \frac{1}{\sqrt{f}}\left\{\pi + \frac{n(a-b)}{6} + \frac{n^2(a^2+b^2)}{24}\left(\frac{\pi}{2} - \frac{4}{3}\right)\right\}.$$

It has already been seen that

$$b = a - \frac{4ka^2}{3} = a - \frac{2na^2}{3};$$

hence, the time of an oscillation

$$= \frac{1}{\sqrt{f}}\left\{\pi + \frac{n^2 a^2}{9} + \frac{n^2 a^2}{12}\left(\frac{\pi}{2} - \frac{4}{3}\right)\right\}$$

$$= \frac{\pi}{\sqrt{f}}\left\{1 + \frac{n^2 a^2}{24}\right\} = \frac{\pi}{\sqrt{f}}\left\{1 + \frac{k^2 a^2}{6}\right\}.$$

The quantity f is $\frac{g}{l}$, l being the length of the pendulum; and $\frac{\pi}{\sqrt{f}}$ is the time of an oscillation in a vacuum. Hence it appears, that when the arc described (a) is small, the time of oscillation is the same as in a vacuum very nearly.

Cor. 1. The arcs described become smaller and smaller, in consequence of the resistance, and the times of the smaller oscillations are somewhat shorter.

The excess of the time in the medium above that in the vacuum, is as the square of the arc described*.

Cor. 2. The same things are true for small circular oscillations†.

Prob. V. *A body oscillates in a cycloidal arc: the resistance being small, and varying as any power of the velocity; it is required to find the decrement of the arc of ascent.*

It will be shewn, that if the resistance vary as the n^{th} power of the velocity, the decremental arc will be as the n^{th} power of the arc of descent‡.

Let resistance $= kv^n$; the other denominations as before;

$$\therefore\ v\frac{dv}{ds} = -(fs - kv^n),$$

$$\frac{v^2}{2} = f(a^2 - s^2) + k\int_s v^n.$$

And $v^2 = 2f(a^2 - s^2)$ nearly, since k is small;

$$\therefore\ \int_s v^n = (2f)^{\frac{n}{2}} \int_s (a^2 - s^2)^{\frac{n}{2}},\ \text{nearly}.$$

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* Hence, Cor. 2, to Prop. 27, Book II, of the *Principia*, is erroneous. It is there asserted, that the excesses of the times above those in a vacuum, are as the arcs themselves.

† For the direct proof of this Proposition in the case of circular arcs, see Poisson, *Traité de Mec.* Art. 273.

‡ *Principia*, Book II, Prop. 31. Cor.

$$= (2f)^{\frac{n}{2}} \int_s \left\{a^n - \frac{n}{2} a^{n-2} s^2 + \frac{n(n-2)}{2.4} a^{n-4} s^4 - \&c.\right\}$$

$$= (2f)^{\frac{n}{2}} \left\{a^n s - \frac{n a^{n-2} s^3}{2.3} + \frac{n(n-2) a^{n-4} s^5}{2.4.5} - \&c.\right\} + C;$$

$$\therefore\ v^2 = 2f(a^2 - s^2) + 2k(2f)^{\frac{n}{2}} \left\{a^n s - \frac{n a^{n-2} s^3}{2.3} + \&c.\right\} + C.$$

Now $v = 0$, when $s = a$. And when v becomes 0 again, suppose $s = -a + \delta$, and neglecting powers of δ, and products $k\delta$, we have

$$0 = 2f.2a\delta + 2k(2f)^{\frac{n}{2}} \left\{-a^{n+1} + \frac{n a^{n+1}}{2.3} - \&c.\right\};$$

$$\therefore\ \delta = 2ka^n (2f)^{\frac{n}{2}-1} \left\{1 - \frac{n}{2.3} + \frac{n(n-2)}{2.4.5} - \&c.\right\}.$$

And hence, δ is as a^n.

Cor. If δ be known, k may be found*.

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* *Principia*, Prop. 30. Cor.

CHAP. IV.

INVERSE PROBLEMS RESPECTING THE MOTION OF POINTS ON CURVES IN RESISTING MEDIA.

195. Here, as in the corresponding Chapter in the first Book, the curve is to be determined from some property of it which is given. Most of the problems, however, which occur, present considerable difficulties, and lead to very complicated calculations: we shall therefore take only one of the most remarkable and celebrated of the questions of this kind. The solution is nearly that given by Laplace, *Mecanique Celeste*, Prem. Par. Liv. I, No. 12.

Prop. *To find the tautochronous curve in a medium, of which the resistance varies partly as the velocity, and partly as the square of the velocity; the body being acted on by gravity.*

In fig. 46, let the vertical abscissa from the lowest point $AM = x$, $AP = s$; the velocity $= v$, and the resistance $= 2hv + kv^2$. The resolved part of gravity in the curve will be $g \cdot \frac{dx}{ds}$, and hence, making t the independent variable, we have for the descent

$$\frac{d^2 s}{dt^2} = -g \cdot \frac{dx}{ds} + (2hv + kv^2);$$

or since $v = -\frac{ds}{dt}$,

$$\frac{d^2 s}{dt^2} + g \cdot \frac{dx}{ds} + 2h \cdot \frac{ds}{dt} - k\frac{ds^2}{dt^2} = 0 \ldots\ldots\ldots (1).$$

Similarly, in the ascent

$$\frac{d^2 s}{dt^2} + g\frac{dx}{ds} + 2h\frac{ds}{dt} + k\frac{ds^2}{dt^2} = 0 \ldots\ldots (2),$$

s being positive.

To integrate equation (1). Let $s = -\frac{1}{k}\,\mathrm{l}.(1 - z)$;

$$\therefore \frac{ds}{dt} = \frac{1}{k}\cdot\frac{1}{1-z}\cdot\frac{dz}{dt};$$

$$\frac{d^2 s}{dt^2} = \frac{1}{k}\cdot\frac{1}{(1-z)^2}\cdot\frac{dz^2}{dt^2} + \frac{1}{k}\cdot\frac{1}{1-z}\cdot\frac{d^2 z}{dt^2}.$$

Substituting these values, we have

$$\frac{1}{k}\cdot\frac{1}{1-z}\,\frac{d^2 z}{dt^2} + gk(1-z)\frac{dx}{dz} + \frac{2h}{k}\,\frac{1}{1-z}\cdot\frac{dz}{dt} = 0;$$

$$\therefore \frac{d^2 z}{dt^2} + 2h\frac{dz}{dt} + k^2 g(1-z)^2\cdot\frac{dx}{dz} = 0 \ldots\ldots (3).$$

Now, suppose the last term of this equation to be expanded in a series of powers of z. It will not involve any power of z below z^1; for at A, where $s = 0$, and therefore $z = 0$, we must have

$$x \propto s^2, \text{ and } \therefore x \propto z^2; \ \therefore \frac{dx}{dz} \propto z.$$

Let therefore

$$k^2 g(1-z)^2\frac{dx}{dz} = Az + Bz^\beta + \&\text{c. where } \beta > 1 \ldots\ldots (4);$$

$$\therefore \frac{d^2 z}{dt^2} + 2h\frac{dz}{dt} + Az + Bz^\beta + \&\text{c.} = 0.$$

Let T be the time of descending to the lowest point, and put $T - t = t'$. Then our equation becomes

$$\frac{d^2 z}{dt'^2} - 2h\frac{dz}{dt'} + Az + Bz^\beta + \&\text{c.} = 0 \ldots\ldots\ldots (5).$$

If we omit the terms after Az, we can integrate this equation, and find a factor which will make the first part of the expression immediately integrable. This factor is

$$\epsilon^{-t'(h-\gamma\sqrt{-1})} \text{ where } \gamma = \sqrt{(A-h^2)},$$

for if we differentiate

$$\epsilon^{-t'(h-\gamma\sqrt{-1})}\left\{\frac{dz}{dt'} - (h+\gamma\sqrt{-1})z\right\},$$

we find

$$\epsilon^{-t'(h-\gamma\sqrt{-1})}\left\{\frac{d^2z}{dt'^2} - 2h\frac{dz}{dt'} + (h^2+\gamma^2)z\right\}dt'.$$

Hence, multiplying equation (5), by $\epsilon^{-t'(h-\gamma\sqrt{-1})}$, and integrating, we have

$$\epsilon^{-t'(h-\gamma\sqrt{-1})}\left\{\frac{dz}{dt'} - (h+\gamma\sqrt{-1})z\right\}$$
$$+ B\int z^{\beta}dt'\epsilon^{-t'(h-\gamma\sqrt{-1})} + \&c. + C = 0.$$

Putting for $\epsilon^{t'\gamma\sqrt{-1}}$ its value $\cos. t'\gamma + \sqrt{(-1)} . \sin. t'\gamma$, and taking the impossible parts, we find

$$\epsilon^{-t'h}\left\{\left(\frac{dz}{dt'} - hz\right)\sin. t'\gamma - \gamma z\cos. t'\gamma\right\}$$
$$+ C' = -B\int_{t'} z^{\beta}\sin. t'\gamma\,\epsilon^{-t'h} + \&c.$$

But when $t'=0$, $t=T$, $s=0$; $\therefore$ $z=0$; and $\therefore$ $C'=0$, the integral on the right-hand side being taken from $t'=0$. Also when $t'=T$,

$$\frac{ds}{dt'} = 0, \text{ and } \frac{dz}{dt} = 0.$$

Let, at that point $z_i = Z$; and we have

$$\epsilon^{-Th}\{h\sin.\gamma T + \gamma\cos.\gamma T\}Z = B\int_{t'} z^{\beta}\sin.\gamma t'\epsilon^{-ht'} + \&c.$$

the integral being taken from $t'=0$, to $t'=T$.

Now, when the oscillation is indefinitely small, and therefore z indefinitely small, the second side vanishes (for $\beta > 1$). Hence, we have in this case,

$$h \sin. \gamma T + \gamma \cos. \gamma T = 0, \ \tan. \gamma T = -\frac{\gamma}{h} \ldots\ldots\ldots (6).$$

But the time T is, by the supposition of tautochronism, independent of the arc, and we must therefore have the same equation for it, when z is not indefinitely small. Hence, we must have

$$0 = B \int_{t'} z^{\beta} \sin. \gamma t' \epsilon^{-kt'} + \&c.$$

between the limits $t' = 0$, and $t' = T$. But this cannot be the case except $B = 0$. For the factor $\sin. \gamma t' \epsilon^{-kt'}$ is always positive between the limits; and when z is small, the first term may represent the whole expression. Hence, the integral cannot be $= 0$; and therefore we must have $B = 0$.

Hence, we have from (4),

$$k^2 g (1 - z)^2 \frac{dx}{dz} = Az;$$

or, since

$$1 - z = \epsilon^{-ks}, \text{ and } \frac{dz}{ds} \frac{1}{(1 - z)^2} = k\epsilon^{ks},$$

$$kg\epsilon^{-ks} \cdot \frac{dx}{ds} = A(1 - \epsilon^{-ks}),$$

$$kg \frac{dx}{ds} = A(\epsilon^{ks} - 1).$$

COR. 1. If we expand ϵ^{ks}, divide by k, and then make $k = 0$, we have

$$g \frac{dx}{ds} = As,$$

which is the expression for the tautochron in a non-resisting medium.

Cor. 2. The expression for T in (6), does not involve k, the coefficient of the square of the velocity. Also, if the resistance involved terms $mv^3 + nv^4 +$ &c. this expression would be the same.

Cor. 3. For the arc of ascent we may proceed in the same manner, making

$$s = \frac{1}{k}\,\mathrm{l}\,(1 + z)\text{; and we shall have}$$

$$kg\frac{dx}{ds} = A(1 - \epsilon^{-ks}).$$

For the brachystochron in a resisting medium, see Woodhouse's *Isoperimetrical Problems*, p. 141.

BOOK III.

THE MOTION OF A RIGID BODY OR SYSTEM.

INTRODUCTORY REMARKS.

196. WHEN a body moves so that in two successive positions of the body, the line joining two successive positions of any point, is equal and parallel to the line joining the two successive positions of the centre of gravity, the motion is called *a motion of translation;* the body is said to move *retaining its parallelism.*

When a body moves, so that a certain line in the body occupies the same situation in space, in two successive positions of the body, the body is said to *revolve* about this line: and this line is called the *axis of revolution:* such a motion is called an angular motion, or *motion of rotation.*

197. PROP. *If a point in a rigid body occupy the same situation in two successive positions of the body, the body may move out of one of the positions into another by revolving round some line passing through the fixed point.*

Let C be the fixed point, and let P, Q, fig. 49, be any two points of the body in its first position, and p, q, the same two points in the next successive position. Since the body is rigid, all the points in the plane CPQ are in that plane in both positions of the body; for the same reason, all the points in the plane Cpq are in that plane in both the positions of the body. If therefore the line CO be the intersection of the planes CPQ, Cpq, all the points in the line CO in one position of the body will be in that line in the other position of the body; and therefore, the body may move out of one of these positions into the other by revolving above the line CO.

198. Prop. *Any motion whatever may be resolved into a motion of translation and a motion of rotation.*

Let C be the situation of the centre of gravity, and P, Q, the situation of two other points in one position of the body; and let c, p, q be the situation of these points in the other position of the body; let the body move *parallel to itself* so that C comes to c; and let p', q' be the positions to which P, Q are thus brought. Then by last Article the body may move out of the position c, p', q' into the position c, p, q by a motion of rotation. Therefore the motion which was supposed to be any whatever, may be resolved as was asserted.

199. When a body revolves continually about the same fixed line, this is called a *permanent axis*. This axis may be merely a mathematical or imaginary line: the points where it meets the surface of the body are called *poles*.

A body, while revolving, may change its axis of rotation, so as to revolve first about one line, and then about another. And we may suppose this change of the axis to be perpetually and gradually taking place; so that the body revolves about the same axis only for an indivisible instant, and the next moment proceeds to revolve about an axis immediately contiguous to the former, and so on continually; the poles of the axis moving perpetually upon the surface of the body. In this case, the axis at any instant is called the *momentary* or *instantaneous axis of rotation*.

CHAP. I.

PRINCIPLES OF THE MOTION OF A RIGID BODY.

200. The conclusions of the preceding part of the Work are true of the motion of a *point:* some of them are also true of a body of finite magnitude, if we suppose it to move, retaining its parallelism: but they are no longer necessarily true if the body have any *rotatory* or *angular motion*. To these cases our mechanical principles must be applied by the aid of new considerations.

When a solid body has any rotatory motion, its parts act upon one another, and thus modify the effects of the other forces by which they are acted on. Any body, under such circumstances, may be considered as a machine; and by means of its rigidity or its other properties, the forces which are applied to one part of the body propagate their effect to another; so that each particle both presses with its own force, and serves to form levers and rods by which the pressures of other particles are communicated. The laws according to which this connexion of different particles modifies the effect of the forces which move them, are to be the subject of our consideration in the present division of the Work.

201. The principles upon which our reasonings must depend are the laws of motion, which have been already applied to points. In applying them to the cases in question, we use the following considerations.

Instead of the forces which act upon any body in motion, *we may substitute equivalent forces according to the principles of Statics.*

Thus a force *P* (*Pp*, fig. 50.) acting at a certain distance (*CP*) to turn a body round an axis (*C*), exerts the same effort as a force 2*P* (*P′p′*) acting similarly at half the radius (*CP′*).

This follows necessarily from the nature of material connexion as a mode of transmitting pressure. When we say that a moving body is rigid, we imply that any force applied at one part communicates its effect to other parts according to the same law as if the body were at rest.

Thus any force which acts to turn a body round an axis, acts effectively upon all the particles; the body itself transmitting the action after the manner, and according to the laws, of a lever.

202. THIRD LAW OF MOTION EXTENDED. *When pressure, transmitted by any material system, produces motion, the moving force is proportional to the pressure.*

The third law of motion, as explained in the Mechanics, asserts that the moving force is as the pressure, in cases of *direct* action; it is here asserted that the same rule applies to *indirect* action, produced by means of any machine on material combination whatever. The moving force is measured by the momentum generated in a given time, as one second.

The proof of the law in this case must be of the same nature as the proof of it in the case of direct action. *In the first place* this is the simplest law which we can conceive. If two equal pressures acting together in a similar manner did not produce a double momentum, it would follow that the action of the second pressure is different when it acts alone, and when it is combined with another pressure; and if this were so, there must be a law which should determine this difference, in addition to the other laws of motion; whereas if the double pressure produce a double momentum, each pressure produces the same effect as if it acted independently, and no additional law is necessary.

In the second place, there is no such difference *obvious*, in common phenomena. Two men can turn a wheel twice as fast as one can, making allowance for the additional force expended in moving their own limbs when the speed increases. A very small overbalance of weight in a machine produces a very small velocity; with the amount of overbalance the velocity increases; a very great overbalance produces a velocity approaching to that of a body falling freely. There are no common and general facts which contradict the assertion that the velocity produced in a given mass in a given time, is as the pressure which produces it. Hence if this assertion be not exactly true, its falsity must be detected by experiments conducted with precise measures. And

In the third place, it is found that in such experiments, the facts agree with the results of calculation on the assumption of the truth of the third law of motion, thus extended. The experiments of Atwood and Smeaton, referred to as proofs of the third law of motion (*Mechanics*, Art. 123), were, in fact, examples of the action of weights *transmitted* by means of certain machinery; and therefore prove the law in the extended sense here intended.

203. The distinction of *impressed* and *effective* forces in a moving body or system has already been spoken of, Article 154. The impressed forces are extraneous pressures; the effective forces are moving forces inferred from the momentum generated in each part of the system in a unit of time; both kinds of forces may be measured as moving forces, that is, as mass multiplied into accelerating force, or as mass multiplied into velocity generated.

The third law of motion, extended as above, may therefore be expressed by stating that *the impressed and effective forces are statically equivalent to each other*. This proposition has already been shewn to be true in the case treated of in the first book; namely, the case of several points moving so that their mutual actions are in the lines joining them. It now appears that it is true likewise in the cases which are there excluded, in which one point produces upon another an effect transverse to the line joining them.

204. The above law of motion is often enunciated and applied in the following form; in which form it is called D'ALEMBERT'S *Principle*.

PROP. *When any forces produce motion in any material system, the moving forces lost by the different parts of the system must balance each other.*

Let m be one of the material points of the system, and let u be the velocity which the force that acts upon m would communicate to it in a unit of time if m were detached: then, if τ be a very small time, $u\tau$ would be the velocity which the force would communicate to m in the time τ. Let $q\tau$ be the velocity which in the actual state of the system is communicated to m in the time τ: and let the velocity $u\tau$ be resolved into two velocities $p\tau$ and $q\tau$. The accelerating force which would produce the velocity $u\tau$ in the time τ, or u in the unit of time, is u; therefore the moving force which would produce this velocity is mu. In like manner the moving forces which would produce the velocities $p\tau$ and $q\tau$ in the time τ, are mp and mq respectively: and since forces are resolved in the same manner as velocities, the force mu may be resolved into mp and mq. Of these forces, mq is effective, and mp is *the force lost* by the point m.

If in the same way m', u', p', q' represent the quantities analogous to m, u, p, q for another point; m'', u'', p'', q'', for another, and so on; $m'p'$, $m''p''$, and so on, will be the forces lost by these points m', m'', &c.

By the last Article all the impressed forces

$$mu,\ m'u',\ m''u'',\ \&c.$$

are equivalent to the effective forces

$$mq,\ m'q',\ m''q'',\ \&c.$$

according to the statical conditions of equilibrium of the system.

That is,

$$mp,\ mq,\ m'p',\ m'q',\ m''p'',\ m''q'',\ \&c.$$

are equivalent to

$$mq,\ m'q',\ m''q'',\ \&c.$$

Therefore the forces

$$mp,\ m'p',\ m''p'',\ \&c.$$

are statically equivalent to nothing; that is, they balance each other on the machine.

Cor. In this case some of the forces lost will be negative, and if the negative sign be omitted, these may be considered as *forces gained.*

Thus if a point m' be acted upon by no forces, $m'q'$ being the effective force, $m'p'$ is $-m'q'$; and the force lost being $-m'q'$, the force gained is $m'q'$.

If we adopt this term, the proposition may be stated by saying that in every system the moving forces lost and gained by the different points balance each other.

CHAP. II.

ROTATION ABOUT A FIXED AXIS.

205. We proceed to determine the angular motion produced when forces act upon a body moveable about a fixed axis. We consider the effect in producing motion only: the other effects, of producing pressure upon the axis, and of affecting the motion of the axis when it is moveable, will be investigated afterwards.

PROP. *In a rigid system consisting of any number of points* m, n, p, q, *&c.* fig. 51, *in the same plane, moveable about an axis* C, *perpendicular to that plane, a force* F *acts to turn the system; to find the effective accelerating force on any point.*

Let F be a moving force which acts perpendicularly at an arm Cf. And let M, N, P, &c. be the effective accelerating forces on m, n, p, &c. Therefore Mm, Nn, Pp, &c. are the effective moving forces; and they are perpendicular to CM, CN, CP, &c. because the motion is so.

Hence, we have

Impressed force .... F acting perpendicularly at an arm CF,

Effective forces .... Mm, Nn, Pp, &c. acting perpendicularly at arms Cm, Cn, Cp, &c.

Hence, by Art. 202, and by the general proposition of the lever,

$$F.Cf - M.m.Cm - N.n.Cn - P.p.Cp - \&c. = 0.$$

But since m, n, p, &c. must all move with the same angular velocity round C, their linear velocities must always be as Cm, Cn, Cp, &c. and therefore all alterations of the velocities must be in this ratio, and the accelerating forces which produce them in the same ratio. Hence, we have $M : N :: Cm : Cn$;

therefore $N = \frac{M.C\overset{n}{\cancel{m}}}{Cm}$; similarly $P = \frac{M.Cp}{Cm}$, &c. And substituting these values in the above equation, it becomes

$$F.Cf - M.m.Cm - M.n.\frac{Cn^2}{Cm} - M.p.\frac{Cp^2}{Cm} - \&c. = 0.$$

Whence $M = \frac{F.Cf.Cm}{m.Cm^2 + n.Cn^2 + p.Cp^2 + \&c.}$ *.

Similarly $N = \frac{F.Cf.Cn}{m.Cm^2 + n.Cn^2 + p.Cp^2 + \&c.}$,

$$P = \frac{F.Cf.Cp}{m.Cm^2 + n.Cn^2 + p.Cp^2 + \&c.},$$

and so on for any other point.

Thus the effective force on $f = \frac{F.Cf^2}{m.Cm^2 + n.Cn^2 + p.Cp^2 + \&c.}$.

The effective force at a distance 1 from the axis

$$= \frac{F.Cf}{m.Cm^2 + n.Cn^2 + p.Cp^2 + \&c.}.$$

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* As this Proposition is the foundation of the whole doctrine of rotatory motion, we shall shew how it may be deduced from elementary laws, independently of the general Principle.

Let F be statically equivalent to a number of accelerating forces, as M at m, N at n, &c. which act perpendicularly to Cm, Cn, &c., and are as Cm, Cn, &c. Therefore

$$F.Cf - M.m.Cm - N.n.Cn \&c. = 0; \text{ and } N = M.\frac{Cn}{Cm}, \&c.$$

$$\therefore F.Cf - M.m.Cm - M.n.\frac{Cn^2}{Cm}, \&c. = 0; \text{ and } M = \frac{F.Cf.Cm}{m.Cm^2 + n.Cn^2} + \&c.$$

Now, by Art. 201, M, N, &c. will produce the same effect as F. Also, if Cm, Cn be supposed to be unconnected and moveable independently about C, and *if the third law of motion be applicable here*, it will follow that when M, N, &c. act on m, n, the accelerations of m, n, will be proportional to Cm, Cn, and therefore the angular velocities of Cm, Cn will be equally increased, and the points m, n of the system will always retain the same position with respect to each other. Hence we may suppose Cm, Cn, &c. to be connected, and the system to become rigid, and the effect will still be the same as when m, n, &c. were unconnected. Hence, when F acts on the rigid system, the effective force in m is the same value of M which we have just found, and which agrees with the value in the text.

If instead of the force F we have any forces acting in any manner upon the system, we must substitute instead of $F.Cf$ the *moment* of these forces about the axis C, that is, the sum of each into the perpendicular upon it from C; those being taken negative which tend to turn the system in the opposite direction.

Cor. 1. Since the effective accelerating force on f is

$$=\frac{F}{m.\frac{Cm^2}{Cf^2}+n.\frac{Cn^2}{Cf^2}+p.\frac{Cp^2}{Cf^2}}.$$

It appears, that the resistance which m opposes to the communication of motion, is the same as that of a mass $m.\frac{Cm^2}{Cf^2}$ placed at f, and acted upon immediately; and similarly of the other particles.

Cor. 2. It appears by the demonstration, that the effective forces on different points are as their distances from the axis C.

Cor. 3. If the force F, and the radius Cf be constant, the effective force on each point will be constant; the motions will be uniformly accelerated, and the formula for such motions may be applied. If F be variable, the formulæ for variable motions may be applied.

Cor. 4. If the force which acts be the *weight* of any body, this body must be included among the bodies m, n, p, &c. in the denominator.

Thus if a system of material points in horizontal planes, m, n, p, fig. 52, be moved about a vertical axis AC, by a weight W acting perpendicularly at the radius Cf, by means of a string passing over a pully B; W moves with the same velocity as a body at the extremity of the area Cf; and therefore the same effective force is employed in moving W, as if it were at f. Hence, we have

$$\text{effective force on } f=\frac{\text{weight of } W.Cf^2}{W.Cf^2+m.Cm^2+n.Cn^2+\&\text{c.}},$$

and the effective force on W is the same.

Cor. 5. The quantities W, m, n, &c. in the denominator in the last Corollary, are the *masses* of the bodies; the weight of W in the numerator is a *moving force*. If g represent the accelerating force of gravity, the weight of W is Wg.

Cor. 6. If the lines m, n, p, &c. be not in the same plane perpendicular to the axis, let lines Cm, $C'n$, $C''p$, &c. be their perpendicular distances from the axis: the same formula will be true, putting these lines for Cm, Cn, Cp, &c.

Or, if we take a plane Cmn, perpendicular to the axis, and refer the points of the system to this plane, by lines parallel to the axis; m, n, p, &c. being the points as thus referred, the same formulæ will be true.

Cor. 7. If, in fig. 51, the body p be not fixed to the radius Cp, but be fastened to the extremity of a flexible string which is perpendicular to Cp at p, and which is kept stretched by the forces which act, the effect, for an instant, will be the same as if p were fixed at the extremity of Cp. For the moving force which the string exerts on the body will be the same as the reaction which it exerts upon the extremity of the radius Cp; and will be for an instant, perpendicular to Cp. Hence the effective moving force on the body will be identical with the force which must act at the extremity of Cp in order to balance the reaction of the string; and the equilibrium of the impressed and the effective forces will still subsist, including the effective force on the body fastened to the string.

Cor. 8. Hence, if as in fig. 71, a body P be connected with the system by a string which is wrapped round a cylinder having the centre of motion C for its centre, this article may be applied, in the same manner as if P were fixed to the cylinder at the point where the string leaves it: for the string will be *at every instant* perpendicular to the radius drawn to the point of its contact with the cylinder.

206. The denominator of the fractions which express the effective forces in the preceding formulæ, is *the sum of each particle multiplied into the square of its distance from* *the axis.* This sum is called the ***Moment of Inertia*** with respect to this axis*. It is a quantity which occurs perpetually in considering the subject of rotation.

If the system, instead of consisting of distinct material points, be a continuous body of finite magnitude, the moment of inertia will be the sum of *each* point into the square of its distance from the axis, and will consist of an indefinite number of terms. The sum of these terms may be found by the integral calculus, as will be shewn in the following Chapter.

If the points be m, m_1, m_2, m_3, &c. and their distances from the axis, Cm, Cm_1, Cm_2, Cm_3, &c. the moment of inertia may be represented by $\Sigma(m \,.\, Cm^2)$. And if F be a moving force which acts perpendicularly at a distance Cf, we shall have the accelerating force at the point where the force acts

$$= \frac{F \,.\, Cf^2}{\Sigma(m \,.\, Cm^2)}.$$

We may suppose k to be such a quantity that

$$k^2 \Sigma m = \Sigma(m \,.\, Cm^2),$$

Σm indicating the sum of all the points of the system.

In this case k is called the *radius of gyration.*

If forces act upon every point of the system, the effect may be calculated by the same principle as before, as will be seen in the next Problem.

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* *The inertia* of a body is the measure of its effect in resisting the communication of motion: in a single point, it is as the mass simply; but in a body revolving about an axis, the effect of a particle in resisting motion depends on the distance from the axis, like the effect of the force acting on a lever. The effect on a lever is as the product of the force and distance, and this product is called *the moment;* the effect of the inertia of the mass in resisting rotatory motion, appears from the above investigation to be as the product of the mass and square of the distance, and hence, this product is called *the moment of inertia:* and the sum of these products is called the moment of inertia of the system.

207. **Prop.** *A rigid system of material points, moveable about a horizontal axis, has all its parts acted on by gravity; it is required to determine the accelerating force.*

Let C, fig. 53, be the axis, and m, n, p, the points. Draw a horizontal line through C, meeting vertical lines through m, n, p, in d, e, h. Then the moving forces impressed are the weights of m, n, p. Let M be the effective accelerating force upon m; therefore in the same way as before, the effective accelerating forces on n, p, are

$$\frac{M.Cn}{Cm}, \quad \frac{M.Cp}{Cm}.$$

And the effective moving forces are

$$M.m, \quad \frac{M.n.Cn}{Cm}, \quad \frac{M.p.Cp}{Cm}.$$

Now Cd, Ce, Ch are perpendicular to the directions of the former forces, and Cm, Cn, Cp are perpendicular to the directions of of the latter forces; also if m be the mass of one of the bodies, its weight or moving force is mg, and so for the rest. Hence, by the principle of the equilibrium between the impressed and the effective forces, Art. 202, and by the property of the lever, we have,

$$m.g.Cd + n.g.Ce + p.g.Ch$$
$$= M.m.Cm + \frac{M.n.Cn^2}{Cm} + \frac{M.p.Cp^2}{Cm};$$

$$\therefore\ M = \frac{(m.Cd + n.Ce + p.Ch)\,Cm.g}{m.Cm^2 + n.Cn^2 + p.Cp^2}.$$

Cor. 1. If we had supposed that there were more bodies, we should have had a corresponding number of terms, both in the numerator and denominator.

Cor. 2. There will be negative terms in the numerator, when any of the bodies are on the other side of the vertical line drawn through C; the terms in the denominator will always be positive, because the bodies all move in the same direction round C; and therefore the effective accelerating forces are always in the same direction.

Cor. 3. The effective accelerating force on any other point of the system, as n, will be

$$N = \frac{(m\,.\,Cd + n\,.\,Ce + p\,.\,Ch)\,Cn\,.\,g}{m\,.\,Cm^2 + n\,.\,Cn^2 + p\,.\,Cp^2}.$$

Cor. 4. If G be the centre of gravity of the system, and if a perpendicular from G meet Ch in H, we have

$$(m\,.\,Cd + n\,.\,Ce + p\,.\,Ch) = (m + n + p)\,.\,CH;$$

and if θ be the angle which CG makes with the vertical,

$$CH = CG\,.\,\sin.\,\theta.$$

Hence,

$$M = \frac{(m + n + p)\,CG\,.\,\sin.\,\theta\,.\,Cm\,.\,g}{m\,.\,Cm^2 + n\,.\,Cn^2 + p\,.\,Cp^2};$$

or, denoting $m + n + p$ by Σm, and the denominator by $\Sigma\,(m\,.\,Cm^2)$, whatever be the number of bodies,

$$M = \frac{Cm\,.\,CG\,.\,g\,\sin.\,\theta\,.\,\Sigma m}{\Sigma\,(m\,.\,Cm^2)}.$$

208. Prop. *To find a point of a rigid system moveable about a fixed horizontal axis, which shall be accelerated by gravity exactly as much as a single point, moveable about the same axis would be accelerated in the same position.*

If O be any point in CG, we shall have

$$\text{accelerating force on } O = \frac{CO\,.\,CG\,.\,g\,.\,\sin.\,\theta\,.\,\Sigma m}{\Sigma\,(m\,.\,Cm^2)}.$$

Now, if a single particle were placed in O, and all the rest removed, we should have

$$\text{accelerating force on particle in } O = g\,\sin.\,\theta.$$

And we have to find O, so that these accelerating forces may be equal. For this purpose, we must have

$$CO . CG . \Sigma m = \Sigma (m . Cm^2);$$

$$\therefore\ CO = \frac{\Sigma . (m . Cm^2)}{CG . \Sigma m} \text{ .... } (a).$$

The point O is called the *Centre of Oscillation;* a single point placed in O, would, in any position of CG, be acted on by the same accelerating force as when O is a point in the system; and therefore, the oscillations of CO and of the system would be exactly the same as if we had but one particle O.

COR. 1. The time of oscillation of the system, is the same as that of a simple pendulum, whose length is CO. Hence, if we make $CO = l$, we shall have the time of one of the small oscillations $= \pi \sqrt{\frac{l}{g}}$.

COR. 2. When we know the moment of inertia, and the place of the centre of gravity, the centre of oscillation with respect to the axis C is found by the formula

$$CO = \frac{\Sigma (m . Cm^2)}{CG . \Sigma m};$$

and this is applicable, whether the system consist of distinct points, or of finite bodies.

CHAP. III.

MOTION OF MACHINES.

209. We shall in the present Chapter apply the preceding principles to determine the motion of the mechanical powers, and of other simple combinations of bodies acting on each other.

Sect. I. Motion about a fixed Axis.

Prop. *To determine the motion of weights on a wheel and axle*, fig. 71.

Let P draw up Q by means of strings wrapping round two cylinders A, B, which have a common horizontal axis. Let a, b, be the radii of the cylinders respectively; and as in Art. 206, let the moment of inertia of the machine AB, about its axis, be $k^2\Sigma m$, or $k^2 M$, M here indicating the whole mass of the machine, and k its radius of gyration. We shall then have

impressed forces, Pg at distance a, $-Qg$ at distance b;

of which the moment is $Pga - Qgb$.

Hence, by Cor. 8, Art. 205, and Cor. 2, Art. 207, we have

$$\text{accelerating force on } P = \frac{(Pa - Qb)\,ga}{Pa^2 + Qb^2 + Mk^2} \text{ downwards;}$$

$$\text{accelerating force on } Q = \frac{(Pa - Qb)\,gb}{Pa^2 + Qb^2 + Mk^2} \text{ upwards:}$$

and since these are constant, the motion may be found by the formulæ for constant forces.

Cor. 1. If $Qb > Pa$, the force will act in the opposite direction, and if Q be at rest originally, it will descend.

Cor. 2. If Tg be the tension of the string by which P hangs, P is impelled downwards by its weight, and upwards by the tension. Hence, the moving force on P is $Pg - Tg$, and the accelerating force $\frac{(P-T)g}{P}$;

$$\therefore \frac{(P-T)}{P}g = \frac{(Pa - Qb)ga}{Pa^2 + Qb^2 + Mk^2},$$

$$T = \frac{P(Qb^2 + Qab + Mk^2)}{Pa^2 + Qb^2 + Mk^2}.$$

Similarly, if T' be the tension of the string by which Q hangs,

$$T' = \frac{Q(Pa^2 + Pab + Mk^2)}{Pa^2 + Qb^2 + Mk^2}.$$

Cor. 3. The pressure on the centre of motion arising from P, Q, will be the sum of these tensions, and will be counteracted by a reaction equal to this sum, for the forces which act on the machine must be in equilibrium according to the principles of statics:

$$\therefore \text{ pressure on the centre } T + T' = \frac{PQ(a+b)^2 + (P+Q)Mk^2}{Pa^2 + Qb^2 + Mk^2 g}.$$

210. Prop. *To determine the motion of weights acting on a combination of wheels and axles*, fig. 72.

The wheels and axles may act on each other, either by means of teeth, as at D, or by strings passing round both the wheel and the axle, and turning them by friction, as at D'; or in other ways: the mechanical conditions of the problem are the same in all these cases.

Let a, b be the radii of the first wheel, and of its axle, respectively; a', b' the radii of the second wheel and axle, a'', b'' of the third, and so on. Then the impressed moving forces are Pg acting at A, and Qg acting in the opposite direction at B. By Statics, the latter would be counterbalanced by a force at P equal to $Qg \cdot \frac{bb'b''}{aa'a''}$. Hence, the moving force impressed, and employed in producing motion, is equivalent to

$$Pg - Qg\frac{bb'b''}{aa'a''} \text{ acting at } A\text{, at distance } a \text{ from } C.$$

Let Mk^2, $M'k'^2$, $M''k''^2$ be the moments of inertia of the respective wheels M, M', M'' about their centres, (including the axles): and let x be the effective accelerating force on P or on A. Then, since the accelerating forces are as the velocities, the accelerating force at D or E will be $\frac{bx}{a}$; that at D' or E' will be $\frac{bb'x}{aa'}$; and that at B or Q, $\frac{bb'b''x}{aa'a''}$.

Now, since the effective accelerating force at P is x, that at any distance r from C, in the wheel M, is $\frac{rx}{a}$; and if m be a particle at that distance, the effective moving force is $\frac{mrx}{a}$. And this is equivalent, in its moment round C, to a force $\frac{mr^2x}{a^2}$, acting at A. And hence, the whole effective moving forces in M are equivalent to a force $\frac{x\Sigma . mr^2}{a^2}$ acting at A; that is, they are equivalent to $\frac{x . Mk^2}{a^2}$ at A.

Similarly, the effective moving forces in M' are equivalent to $\frac{bx}{a} . \frac{M'k'^2}{a'^2}$ at E; which force is, by the property of the machine, equivalent in its effect to turn the system round C, to a force $\frac{b^2x}{a^2} . \frac{M'k'^2}{a'^2}$ at A.

The effective accelerating force on Q is $\frac{bb'b''x}{aa'a''}$; which gives a moving force $\frac{bb'b''x}{aa'a''}Q$, equivalent to $\frac{b^2b'^2b''^2x}{a^2a'^2a''}$ at A.

Hence, we have the moment of the impressed forces, about C,

$$= ga \left(P - Q\frac{bb'b''}{aa'a''}\right) .$$

And the moment of all the effective forces, about C,

$$= Pax + Qax \frac{b^2 b'^2 b''^2}{a^2 a'^2 a''^2}$$

$$+ Max \, . \, \frac{k^2}{a^2} + M'ax \, . \, \frac{b^2}{a^2} \, . \, \frac{k'^2}{a'^2} + M''ax \, . \, \frac{b^2 b'^2}{a^2 a'^2} \, . \, \frac{k''^2}{a''^2} \, .$$

Equating these forces (by Art. 202,) and putting n, n', n'' for $\frac{b}{a}$, $\frac{b'}{a'}$, $\frac{b''}{a''}$, we have, for the accelerating force on P,

$$x = \frac{(P - Qnn'n'')g}{P + Qn^2 n'^2 n''^2 + M\frac{k^2}{a^2} + M'n\frac{k'^2}{a'^2} + M''nn'\frac{k''^2}{a''^2}} \, .$$

The accelerating force on $Q = nn'n''x$. These accelerating forces are constant.

211. A machine was constructed by Atwood, to measure the spaces and velocities of bodies descending by gravity, in order to compare them with theory. It is represented in fig. 73. Two equal weights P, Q, hang by a fine string over a fixed pully M. One of them is made to descend, by placing upon it a small weight D, and the times and spaces of the motion are observed. The weights at P and Q are inclosed in equal and cylindrical boxes; so that the effect of the resistance of the air will be the same upon both, and will not affect the motion. And the effect of friction is nearly removed by making the axis of M very slender, and causing each end of it as C to rest upon *Friction Wheels*, as M, M'*. The

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* To shew that these wheels will diminish the effect of friction, we may consider friction as a force acting in a tangent to the axle. If the axle C rested on immoveable surfaces, and the amount of the friction were F, its effect at A would be $\frac{Fb}{a}$. But if the axle C rest upon friction wheels, their circumferences will turn with the circumference of the axle, and between these surfaces there will be no friction. The friction will take place at the axles C, C'; and if we suppose it to be F at the surface of the axle C', this force will be equivalent to $F\frac{b'}{a'}$ at the circumference of the axle C, and to

times are observed by means of a pendulum, and the spaces by a scale of inches *BF*, down which *P* descends. To determine the velocity, *P* is made to pass through an opening *MN*, in a stage fastened to the scale *BF*; and the weight *D*, which is too large to pass, is left resting on *M*, *N*. Therefore, since the weight of *D* produces the only accelerating force which exists in the machine, after passing the point *E*, *P* will move uniformly with the velocity acquired. When *P* has passed through a given space *EF*, it is stopped by striking the stage *F*, which is there fixed to the scale.

The body *P* being allowed to descend from rest at a given point *B*, descends till *D* is heard to strike the stage *M*, *N*, and the time is noted; it then descends till *P* is heard to strike the stage *F*, and the time is noted: the space *EF*, divided by the interval of the two times, gives the velocity; and the space *BF*, and the time of describing it, being known, we can compare our results with theory. The velocities of *P*, *Q* are small, both because *D* is small, and because the wheels *F*, *M'*, *M''* are to be moved, and their moment of inertia is a part of the denominator of the accelerating force.

Observing, that besides the friction wheels, *M'*, *M''*, there are two others at the other end of the axis *A*; calling the moment of each of these $M'k'^2$, and that of *M*, Mk^2, and the radii of the wheels and axles a, b, a', b', we have

$$\text{accelerating force on } P = \frac{Dg}{2P + D + M\dfrac{k^2}{a^2} + 4M' \cdot \dfrac{b}{a}\dfrac{k'^2}{a'^2}}.$$

The effect of the inertia of the wheels is the same as if a mass $M\dfrac{k^2}{a^2} + 4M' \cdot \dfrac{bk^2}{aa'^2}$ were collected at the circumference of *M*.

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to $F\dfrac{bb'}{aa'}$ at *A*. And as there are four ends of the axles *C'* for one *C*, the effect of the friction, when such friction wheels are used, is $4F\dfrac{bb'}{aa'}$ acting at *A*. Hence, by means of such a contrivance, it is diminished in the ratio of $a' : 4b'$, supposing *F* to be the same in both.

The reader will find in Atwood's *Treatise on Rectilinear and Rotatory Motion*, Sect. 7, an account of experiments made with this machine. The results of all of them agreed accurately with the formulæ for constant forces.

212. PROP. *To determine the motion of weights on a lever*, fig. 74.

Let P, Q, be attached to the extremities of a lever whose arms are a, b; and let M be the mass of the lever, and h the distance of its centre of gravity from the center of motion. Let PQ be any position in which the lever makes an angle θ with the vertical. Then $a \cos. \theta$, $b \cos. \theta$, $h \cos. \theta$ are the perpendiculars from the centre of motion upon the vertical directions of the forces of P, Q, M. And the moment of the forces is $(Pa + Mh - Qb)\, g \cos. \theta$, which acts to make P descend. Hence, if Mk^2 be the moment of inertia of the lever itself, we have

$$\text{accelerating force on } P = \frac{(Pa + Mh - Qb)\, g a \cos. \theta}{Pa^2 + Qb^2 + Mk^2}$$

acting perpendicular to CP.

The velocity of P is

$$-a\frac{d\theta}{dt}; \text{ hence, } \frac{ds}{dt} = -a\frac{d\theta}{dt};$$

and, by the formula $v\dfrac{dv}{ds} = f$, we have, v representing P's velocity,

$$v\frac{dv}{d\theta} = -\frac{(Pa + Mh - Qb)\, g a \cos. \theta}{Pa^2 + Qb^2 + Mk^2}; \text{ and integrating,}$$

$$v^2 = C - \frac{2(Pa + Mh - Qb)\, g a \sin. \theta}{Pa^2 + Qb^2 + Mk^2}.$$

If the lever descend from rest from a position AB, let AB make an angle θ_1 with the vertical, and we have

$$\frac{2(Pa + Mh - Qb)\, g a}{Pa^2 + Qb^2 + Mk^2}(\sin. \theta_1 - \sin. \theta) = v^2 = a^2\frac{d\theta^2}{dt^2}:$$

and hence, by integrating, we should find the relation of θ and t.

If P descend through a quadrant, we have, at the lowest point O,

$$v^2 = \frac{2(Pa + Mh - Qb)ga}{Pa^2 + Qb^2 + Mk^2}.$$

213. Prop. *A body moveable about an axis* C *is* struck *at a given point by a given mass with a given velocity; to determine its motion*, fig. 75.

Impact is, properly speaking, a violent pressure continued for a short time. Now if any force act at a distance a from the axis of a body whose moment of inertia is Mk^2, the effect produced at any instant will be the same as if a mass $\frac{Mk^2}{a^2}$ were collected at the distance a. Hence, the whole effect produced will be the same as if such a mass were substituted for the body, whatever be the time which the change employs. And hence, the effect of perpendicular impact at a distance a will be the same as if it took place upon a mass $\frac{Mk^2}{a^2}$ placed there.

In fig. 75, let a mass P impinge directly (that is, in a direction perpendicular to the surfaces where the contact takes place,) on a system CA, with a velocity V; and let CA be a perpendicular on P's direction. If $CA = a$, the effect on the system will be the same as if P impinged on a body $\frac{Mk^2}{a^2}$. Let the substances be supposed inelastic, and the bodies will both move with the same velocity after the impact; and since, by the third law of motion, the mass multiplied into the velocity will be the same before and after the blow, we shall have, if x be the velocity of A after the stroke,

$$x\left(P + \frac{Mk^2}{a^2}\right) = PV,$$

$$x = \frac{PVa^2}{Pa^2 + Mk^2}.$$

If the body be acted upon by no force after the impact, it will revolve uniformly. If it move about a horizontal axis, and be acted on by gravity, it will ascend till all the velocity be destroyed, and then descend, and so oscillate.

If the bodies be elastic, we must apply the rules for impact in that case. On this supposition, P and M will separate after the impact. And if the impact be not direct, we must, supposing the bodies perfectly smooth, take that part of it which is perpendicular to the surfaces at the point of contact.

214. An instrument depending upon these principles was constructed by Robins for the purpose of measuring the velocities of musquet and cannon bullets, and has been called the Ballistic Pendulum. It consisted in an iron plate CA, fig. 76, suspended from a horizontal axis at C, and faced with a thick board DE. When this was at rest, a bullet was fired into the board as at P, which caused the pendulum to move through an arc MN. The chord of this arc was known by means of a ribbon fastened to the pendulum, as at N, and sliding through a slit at M, so that when drawn to the length MN it did not return. The ball stuck in the wood, and was prevented from going through by the iron.

Let O be the centre of oscillation of the pendulum, including the bullet. Then the motion of the pendulum when left to itself, will be the same as if all the matter were collected in O. And hence the arc up which O will move will be that down which it would acquire the velocity which it has at the lowest point. If θ be this angle, the velocity acquired in describing it would be that acquired down the versed sine of θ; or down a perpendicular height $CO \cdot \text{ver. sin.}\, \theta$. Let $CO = l$; $\therefore$ (velocity)^2 of O at lowest point

$$= \sqrt{(2gl \,\text{ver. sin.}\, \theta)} = 2 \sin. \frac{\theta}{2} \sqrt{(gl)}.$$

But since the velocity of P at the lowest point is, by last Article,

$$\frac{PVa^2}{Pa^2 + Mk^2},$$

the velocity of O, which is to this as CO to CP, will be

$\frac{PVal}{Pa^2 + Mk^2}$, which is $= 2 \sin. \frac{\theta}{2} \surd(gl)$ by what has been said.

If h be the distance from C of the centre of gravity of the pendulum, including the ball, by Art. 208,

$$l = \frac{Pa^2 + Mk^2}{(P + M)h}; \quad Pa^2 + Mk^2 = (P + M)hl;$$

$$\therefore \; PVal = 2 \sin. \frac{\theta}{2} (P + M) hl \surd(gl),$$

$$V = 2 \sin. \frac{\theta}{2} . \frac{P + M}{P} \frac{h}{a} \surd(gl).$$

If the pendulum, after being struck by the ball, makes n oscillations in a minute, we have

$$\text{time of oscillation} = \frac{60}{n} = \pi \sqrt{\frac{l}{g}}; \quad \therefore \surd(gl) = \frac{60g}{\pi n}.$$

$$\text{And, } V = 2 \sin. \frac{\theta}{2} . \frac{P + M}{P} . \frac{60gh}{\pi n a}.$$

This agrees with Dr Hutton's formula. We have $2 \sin. \frac{\theta}{2}$ by dividing the chord MN by the radius CN.

Dr Hutton himself however, in his own experiments, found the velocity of balls, by suspending the cannon which he used, and observing the arc through which it was driven by the recoil. The same formula is still applicable, M now representing the weight of the cannon and its appendages without the ball. For the effect will be the same, whether a velocity be communicated to the pendulous body by the impact of the ball, or by its reaction. And the momentum communicated at the axis of the cannon will be PV, because the *momentum* communicated to the ball in one direction, and to the pendulum in the other, must be equal.

It is found by experiments of this kind, that the velocity of musquet and cannon bullets varies from 1600 to 2000 feet per second.

Sect. II. MOTION OF BODIES ROLLING AND UNROLLING.

215. A body *rolls* when it moves in contact with a line or surface so that the parts of the surface of the body are applied continuously to the successive portions of the line or surface. In this case a motion of translation implies also a motion of rotation, and conversely; and a force which produces the one of these motions must produce the other.

When a string has its end fastened to a body, the body by revolving in a proper direction, wraps the string on or unwraps it off the body, and produces a motion which resembles rolling; when the body thus unwraps itself we may term the motion *unrolling*.

In all such cases the body does not revolve about a permanent fixed axis, and therefore the proportions of the preceding chapter are not here immediately applicable. The following proposition will however enable us to reason concerning such cases.

In all these cases we shall suppose all the motions to take place in planes perpendicular to the axis of revolution: the consequence of this will be that the axis of instantaneous revolution will always be parallel to the axis of apparent revolution passing through the centre of gravity, or through any other point.

216. PROP. *When a body rolls or unrolls, the centre of gravity moving in a straight line, it is required to find the resultant of the effective forces, and their moment.*

Let the body move for two successive seconds, and in each of these seconds let it be supposed that the motion is resolved in the following manner;—the centre of gravity moves out of its initial into its final position, the body moving parallel to itself, and then the body moves out of the position into which it is thus brought into its final position, the centre of gravity remaining at rest. In this manner the motion in each second will be resolved into a motion of translation and a motion of revolution.

The effective force (supposing it to be constant) is measured by the increment of velocity in one second. The velocity of each point being resolved in the manner just described, the increments of the velocity will be resolved in the same manner, and therefore the effective forces will likewise be so resolved. Hence the effective forces will be those which are due to the motion of translation, together with those which are due to the motion of rotation.

If f be the effective accelerating force which is due to the motion of the centre of gravity, mf will be the moving force due to the motion of translation of any particle m; and the sum of all those forces will be $\Sigma . mf$, which, since they are all equal and parallel, will be $f\Sigma m$, or Mf, if M be the mass of the body.

Also the resultant of all the parallel forces mf will pass through the centre of gravity of the body, and will be in the direction of the motion of the centre of gravity: therefore its moment with respect to any point in that direction will be nothing.

The resultant of all the effective forces due to the rotatory motion will be nothing; for if ϕ be the effective accelerating force on a particle at the distance 1 from the axis of rotation passing through the centre of gravity, $mr\phi$ will be the effective moving force on a particle m at distance r from the axis, and will act perpendicular to the radius r. If x, y be the co-ordinates of the point m as referred to the centre of gravity by rectangular axes in the plane of rotation, $my\phi$ and $-mx\phi$ will be the components of $mr\phi$ in the directions of these coordinates; and $\Sigma . my\phi$ and $\Sigma . -mx\phi$ will be the resulting forces in these directions. But $\Sigma . my\phi = \phi\Sigma my = 0$, by the property of the centre of gravity; and in like manner $\Sigma . -mxy = \phi\Sigma mx = 0$. Therefore the resultant of the effective forces due to the rotatory motion is 0.

The moment of the force $mr\phi$ with reference to the axis passing through the centre of gravity is $mr^2\phi$, and the sum of all such moments is $\Sigma . mr^2\phi = \phi\Sigma . mr^2 = Mk^2\phi$, Mk^2 being the moment of inertia for the axis now supposed.

Hence, it appears that the effective forces which act on the system M, are equivalent to a force Mf acting at the centre of gravity, and in the direction of the motion of that centre; and that their moment is equivalent to a moment $Mk^2\phi$, acting to move the body about the point where its centre of gravity is at any instant; f being the effective force due to the motion of the centre of gravity, and ϕ the effective force due to the rotatory motion of a particle at a distance 1 from the axis of rotation passing through the centre of gravity.

217. **Prop.** *A cylindrical body* M (fig. 77.) *unrolls itself from a vertical string* ABP, *which passes over a fixed pully and has a weight* P *appended to its other extremity; to determine the motion.*

The tension of the string ABP will be the same throughout, if we neglect the inertia of the pully B; let this be supposed, and let the tension be Tg; and let, as in the last article, Mk^2 be the moment of inertia, f the effective accelerating force on the centre of gravity, ϕ the effective accelerating force on a particle at a distance 1 from the axis; and let a be the radius of the cylinder.

The impressed forces are Tg at the circumference of the cylinder, acting upwards, and the force of gravity on each of the particles of the body, which is equivalent to Mg at the centre of gravity acting downwards. The impressed and effective forces must have their resultants equal and opposite, and also their moments equal and opposite, by the conditions of statical equilibrium. Therefore, by last article,

$$Mg - Tg = Mf,$$

$$Tga = Mk^2\phi.$$

But if f' be the effective accelerating force which acts on P, we shall have, for a like reason,

$$Pg - Tg = Pf'.$$

Now it is clear that if, when any radius, as CA, is horizontal, the point C moves downwards with a velocity v, and the point A moves upwards with a velocity v', the relative velocity of A with respect to C, will be $v + v'$, and the angular velocity of A about C, will be $\dfrac{v+v'}{a}$.

Hence the effective accelerating force due to the rotatory motion will be

$$\frac{d\dfrac{v+v'}{a}}{dt}, \text{ which is } \frac{1}{a}\frac{dv}{dt} + \frac{1}{a}\frac{dv'}{dt} = \frac{f+f'}{a}.$$

Therefore $\phi = \dfrac{f+f'}{a}$, or $f + f' = a\phi$.

Taking the values of f, f' and ϕ from the above three equations, this gives

$$PMg - PTg + PMg - MTg = PTg\frac{a^2}{k^2}.$$

Hence, $2PMk^2 = \{Pa^2 + (M+P)k^2\}\,T$,

$$T = \frac{2PMk^2}{Pa^2 + (M+P)k^2}.$$

From this we find $f = \dfrac{Mg - Tg}{M}$

$$= \frac{Pa^2 + (M-P)k^2}{Pa^2 + (M+P)k^2}.g$$

$$f' = \frac{Pg - Tg}{P}$$

$$= \frac{Pa^2 - (M-P)k^2}{Pa^2 + (P+M)k^2}g$$

$$\phi = \frac{Tga}{Mk^2}$$

$$= \frac{2Pga}{Pa^2 + (M+P)k^2}.$$

These forces are constant, and therefore the motion of the centre of gravity downwards, the motion of P downwards, and the motion of rotation, will be uniformly accelerated.

Cor. 1. If f be negative, M will ascend, that is, if

$$Pa^2 + (M - P)k^2 < 0, \text{ or if } \frac{M}{P} < \frac{k^2 - a^2}{a^2}:$$

this cannot be, except a is less than k.

Cor. 2. If f' be negative, P will ascend, that is, if

$$Pa^2 - (M - P)k^2 < 0, \text{ or if } \frac{M}{P} > \frac{k^2 + a^2}{a^2}.$$

Cor. 3. It is not necessary that the whole body should be cylindrical, but only that the part of it from which the string unrolls should be a cylinder, of which the axis passes through the centre of gravity. The vertical plane of the string must be perpendicular to the axis of the cylinder, and must pass through the centre of gravity.

Cor. 4. If the figure be a cylindrical shell of small thickness, $k = a$,

$$\text{accelerating force on } C = \frac{Mg}{2P + M},$$

$$\text{accelerating force on } P = \frac{2P - M}{2P + M} \cdot g,$$

$$\text{tension} = \frac{2MP}{2P + M}.$$

Cor. 5. If the figure be a solid homogeneous cylinder, it will be seen in next Chapter that $k^2 = \frac{a^2}{2}$.

$$\text{Hence, accelerating force on } C = \frac{P + M}{3P + M} \cdot g,$$

$$\text{accelerating force on } P = \frac{3P - M}{3P + M} \cdot g,$$

$$\text{tension} = \frac{2MP}{3P + M}.$$

218. **Prop.** ***A cylindrical body unrolls itself from a vertical string, the other end of which is fixed; to determine the motion,*** fig. 78,

If we assume P, in last Article, to be such that it shall neither ascend nor descend, we may suppose the string AB to be fixed at the point B, and the motion will be the same as before. We must therefore in this case, have the accelerating force on $P = 0$;

$$\text{or, } Pa^2 - (M - P)k^2 = 0, \quad \text{whence, } P = \frac{Mk^2}{a^2 + k^2}.$$

$$\text{Hence also, } T = \frac{Mk^2}{a^2 + k^2}.$$

$$\text{And accelerating force on } C = \frac{(M - T)g}{M} = \frac{a^2 g}{a^2 + k^2}.$$

Cor. 1. If the figure be a cylindrical shell, $k = a$;

$$\text{accelerating force on } C = \frac{g}{2}, \quad T = \frac{M}{2}.$$

Cor. 2. If the figure be a solid cylinder, $k^2 = \frac{a^2}{2}$, (see next Chapter);

$$\text{accelerating force on } C = \frac{2g}{3}, \quad T = \frac{M}{3}.$$

Cor. 3. If the figure be a globe, it will appear in next Chapter that $k^2 = \frac{2a^2}{5}$;

$$\text{accelerating force on } C = \frac{5g}{7}, \quad T = \frac{2M}{7}.$$

Cor. 4. If the string, instead of being vertical, be laid along an inclined plane as *BA*, fig. 79, the same conclusions are manifestly true; putting for g the force of gravity down the plane, which is $g \times$ sin. inclination. The tension will also be $Tg \times$ sin. inclination.

Cor. 5. If *M*, instead of rolling by means of a string, roll down the plane in consequence of the friction entirely preventing its sliding, the results will be the same. The tension of the string is now replaced by the effort which the friction exercises to prevent sliding.

Hence, when a body rolls down an inclined plane, the accelerating force is $\frac{1}{2}$ if it be a hollow cylinder, $\frac{2}{3}$ if it be a solid cylinder, and $\frac{5}{7}$ if it be a globe, of the force with which a body would slide down the plane, if friction were removed.

Sect. III. Motion of Pullies.

219. Prop. *One body draws another over a single fixed pully; to determine the motion*, fig. 80.

Let Mk^2 be the moment of inertia of the pully, a its radius. And let x be the effective accelerating force on P downwards; which is therefore also the effective accelerating force on the circumference of the pully M, and the effective accelerating force on Q upwards. Let Tg be the tension of the string AP, and $T'g$ of BQ. Hence, the force impressed at the circumference of the pully is $Tg - Tg'$, and therefore, by Art. 208,

$$x = \frac{(T - T')\,ga^2}{Mk^2} \dots\dots\dots\dots\dots (1).$$

But the accelerating force on $P = x = \dfrac{(P - T)\,g}{P}$;

and the accelerating force on $Q = x = \dfrac{(T' - Q)\,g}{Q}$;

$$\therefore\ Px = (P - T)\,g, \quad Qx = (T' - Q)\,g;$$

$$\therefore\ (P + Q)\,x = (T' - T)\,g + (P - Q)\,g \dots\dots\dots (2).$$

Multiply (1) by Mk^2, and (2) by a^2, and add;

$$\therefore\ Mk^2x + (P+Q)a^2x = (P-Q)ga^2,$$

$$x = \frac{(P-Q)ga^2}{Mk^2 + (P+Q)a^2}.$$

Cor. 1. The tensions of AP, and BQ, are respectively

$$\frac{(Mk^2 + 2Qa^2)Pg}{Mk^2 + (P+Q)a^2},\quad \frac{(Mk^2 + 2Pa^2)Qg}{Mk^2 + (P+Q)a^2}.$$

Cor. 2. Hence, when strings are in motion about pullies, the tension of each string is no longer the same throughout its length. A part of the tension of PA is employed in turning M; and it is only the remainder which is continued along the cord, so as to act in BQ.

The same results might have been obtained from Art. 213, by making the radii of the wheel and axle equal.

220. Prop. *In the single moveable pully with the strings parallel; to determine the motion*, fig. 81.

Let P, Q, be the weights; Mk^2, $M'k'^2$, the moments of inertia of the fixed and moveable pullies AB, DE; a, a' their diameters. And let the tension of $AP = Tg$, of $BD = T'g$, of $EF = t'g$. Then, if x be the accelerating force on P, $\frac{x}{2}$ will be the accelerating force on Q, because it moves with half the velocity. Also, the accelerating force at the circumference of M will be x: and since, while E remains fixed, the centre of M' rises with half P's velocity, the relative motion of E *round the centre*, is half P's velocity, and therefore, the effective accelerating force at the circumference of M' round C' is $\frac{x}{2}$.

And if we consider the forces which act upon M', we have

Impressed forces, $T'g$, $t'g$ upwards, Qg downwards;

Q including the weight of M'.

Effective forces, $\frac{\ddot{x}}{2}$ acting upwards on Q, and $\frac{\ddot{x}}{2}$ acting at the circumference, turning M' round C'; whence the force round C' is $\frac{\ddot{x}}{2a}$, at distance 1, and by Art. 216, the moment round C' is $\frac{\ddot{x}}{2}\,\frac{M'k'^2}{a'}$.

Hence, by Art. 202, we must have

$$(T'+t')g - Qg = Q\frac{\ddot{x}}{2};$$

and, considering the moments with respect to C',

$$(T'-t')\,a'g = \frac{M'k'^2}{a'}\cdot\frac{\ddot{x}}{2};$$

$$\therefore\ 2T'g = Qg + Q\frac{\ddot{x}}{2} + \frac{M'k'^2}{a'^2}\cdot\frac{\ddot{x}}{2}.$$

Also, we have, as before,

$$\ddot{x} = \frac{P-T}{P}g, \text{ or } (P-T)g = P\ddot{x},$$

$$\ddot{x} = \frac{(T-T')ga^2}{Mk^2}, \text{ or } (T-T')g = \frac{Mk^2}{a^2}\ddot{x};$$

add these, and the result of former one, namely,

$$T'g = \frac{Qg}{2} + \frac{Q\ddot{x}}{2^2} + \frac{M'k'^2}{a'^2}\cdot\frac{\ddot{x}}{2^2},$$

and we have

$$Pg = \frac{Qg}{2} + \left(P + \frac{Q}{2^2} + \frac{Mk^2}{a^2} + \frac{M'k'^2}{2^2a'^2}\right)\ddot{x};$$

$$\therefore\ \ddot{x} = \frac{\left(P - \frac{Q}{2}\right)g}{P + \frac{Q}{2^2} + \frac{Mk^2}{a^2} + \frac{M'k'^2}{2^2a'^2}};$$

from this also the tensions might be found.

221. Prop. *In the system of moveable pullies, where each pully hangs by a separate string; to determine the motion*, fig. 82.

The strings are supposed parallel.

Let M, M', M'', M''' be the pullies; Mk^2, $M'k'^2$, $M''k''^2$, &c. their moments; a, a', a'', &c. their radii. Let x be the effective accelerating force on P; then $\frac{x}{2}$ will be the accelerating force on M'; $\frac{x}{2^2}$ on M''; $\frac{x}{2^3}$ on M'''; and these will also be the effective accelerating forces producing rotation at the circumferences of M, M', M'', &c. And, by reasoning with respect to each pully, as we have done for M' in last Article, we have these equations,

$$(P-T)g = Px, \quad (T-T')g = \frac{Mk^2}{a^2}x.$$

$$(T'+t')g - M'g - T''g = M'\frac{x}{2}, \quad (T'-t')g = \frac{M'k'^2}{a'^2}\frac{x}{2},$$

$$(T''+t'')g - M''g - T'''g = M''\frac{x}{2^2}, \quad (T''-t'')g = \frac{M''k''^2}{a''^2}\frac{x}{2^2};$$

and so on.

Eliminating t', t'', &c. from each successive pair, we have

$$Pg = Px + Tg,$$

$$Tg = \frac{Mk^2}{a^2}x + T'g,$$

$$T'g = \frac{M'k'^2}{a'^2}\cdot\frac{x}{2^2} + \frac{M'x}{2^2} + \frac{M'g}{2} + \frac{T''g}{2},$$

$$T''g = \frac{M''k''^2}{a''^2}\cdot\frac{x}{2^3} + \frac{M''x}{2^3} + \frac{M''g}{2} + \frac{T'''g}{2},$$

and so on.

Therefore,

$$Pg = Px + \frac{Mk^2}{a^2}x + \frac{M'k'^2}{a'^2}\cdot\frac{x}{2^2} + \frac{M''k''^2}{a''^2}\cdot\frac{x}{2^4} + \text{\&c.}$$
$$+ \frac{M'x}{2^2} + \frac{M''x}{2^4} + \text{\&c.} + \frac{M'g}{2} + \frac{M''g}{2^2} + \text{\&c.} + \frac{T^{\text{iv}}g}{2^3}.$$

The law of continuation is manifest: and the last tension (at T^{iv} in the figure) is that which immediately raises Q. Hence, we have

$$\text{the effective accelerating force on } Q, = \frac{x}{2^3} = \frac{(T^{\text{iv}} - Q)g}{Q};$$

$$\therefore\ T^{\text{iv}} g = Qg + Q\frac{x}{2^3}.$$

Substituting this, and obtaining the value of x, we have

$$x = \frac{\left(P - \frac{M'}{2} - \frac{M''}{2^2} - \&\text{c.} - \frac{Q}{2^3}\right)}{P + \frac{Mk^2}{a^2} + \frac{M'}{2^2} + \frac{M'k'^2}{2^2 a'^2} + \frac{M''}{2^4} + \frac{M''k''^2}{2^4 a''^4} + \&\text{c.} + \frac{Q}{2^6}};$$

and similarly for any number of pullies.

By similar reasoning, we shall have the accelerating force in the system of pullies in which each is attached to the weight: but more immediately in all such cases by the next Proposition but one.

222. **Prop.** ***The moment of inertia of any system, with respect to any given axis, is equal to the moment of inertia about an axis parallel to this, passing through the centre of gravity, plus the moment of inertia of the whole body, (collected in its centre of gravity,) about the given axis.***

Let fig. 54 represent any system, moveable about an axis C; and let m, n, p, q, be the particles of it, referred to a plane perpendicular to the axis. Let G be the centre of gravity of m, n, p, q. Draw md perpendicular on CG.

$$\text{Now, } Cm^2 = CG^2 + Gm^2 + 2CG \cdot Gd.$$

Similarly, if ne, ph, qk, be perpendicular on CG,

$$Cn^2 = CG^2 + Gn^2 - 2CG \cdot Ge,$$

$$\&\text{c.} = \&\text{c.};$$

$$\begin{aligned}\therefore\ m\,.\,Cm^2+n\,.\,Cn^2+p\,.\,Cp^2+\&\text{c.}\\ =m\,.\,CG^2+n\,.\,CG^2+p\,.\,CG^2+\&\text{c.}\\ +m\,.\,Gm^2+n\,.\,Gn^2+p\,.\,Cp^2+\&\text{c.}\\ +2CG\,(m\,.\,Gd-n\,.\,Ge+p\,.\,Gh-\&\text{c.})\end{aligned}$$

And, since by the property of the centre of gravity,

$$m\,.\,Gd-n\,.\,Ge+p\,.\,Gh-\&\text{c.}=0;$$

we have

$$\begin{aligned}m\,.\,Cm^2+n\,.\,Cn^2+p\,.\,Cp^2+\&\text{c.}=(m+n+p+\&\text{c.})\,CG^2\\ +m\,.\,Gm^2+n\,.\,Gn^2+p\,.\,Gp^2+\&\text{c.}\end{aligned}$$

Or, moment of inertia round C
= moment of inertia of $(m+n+p+\&\text{c.})$
at G round C + moment of inertia round G.

Cor. 1. We may represent this theorem thus, whatever be the number of bodies;

$$\Sigma\,(m\,.\,Cm^2)=CG^2\,\Sigma m+\Sigma\,(m\,.\,Gm^2)\,;$$

or if $Cm=r$, $CG=a$, $\Sigma m=M$, $\Sigma(m\,.\,Gm^2)=Mk^2$,

it will become $\Sigma\,.\,mr^2=M\,(a^2+k^2)$.

Cor. 2. Knowing the moment of inertia round G, we may, from this expression, find the moment round C.

Cor. 3. The moment round G, the centre of gravity, is less than that round any other axis C, parallel to G.

223. Prop. *To find the accelerating force resulting from the action of gravity on any machine whatever, in which the ratios of the velocities of different points are constant.*

Let P be one of the bodies of the machine, and let P' be the mass, which, placed at P, would preserve the equilibrium. Then the weight $(P-P')g$ is the *impressed force*, which produces the motion.

Let u be the velocity of P, and v, v', &c. the velocities of any other bodies m, m', &c. in the system. If x be the effective accelerating force on P, since the forces are in the ultimate ratio of the increments of the velocities $x\dfrac{dv}{du}$ will be the accelerating force on m, and $mx\dfrac{dv}{du}$ the effective moving force. Therefore the forces which must balance each other according to Art. 202, are $(P-P')g$, &c. in one direction, and Px, $mx\dfrac{dv}{du}$, $m'x\dfrac{dv'}{du'}$, &c. in the opposite direction.

Now u, v, v', &c. may be considered as the virtual velocities of the points where these forces are applied. Hence by the principle of virtual velocities,

$$(P-P')g\,.\,u - Px\,.\,u - mx\frac{dv}{du}\,.\,v - m'x\frac{dv'}{du}\,.\,v' - \text{\&c.} = 0\,;$$

$$\therefore\ x = \frac{(P-P')g}{P + \dfrac{mv}{u}\dfrac{dv}{du} + \dfrac{m'v'}{u}\dfrac{dv'}{du} + \text{\&c.}} = \frac{(P-P')g}{P + \dfrac{\Sigma mv}{u}\dfrac{dv}{du}}.$$

Let a mass M, which forms any part of the system, be considered as for an instant moving about some stationary axis. This is always a possible way of representing the motion (see Art. 199.). Let the distance of the center of gravity of M from the imaginary stationary axis be a, and the velocity of this center be α. Then if m be any particle of M which is at the distance r from the axis, m's velocity $= \dfrac{r\alpha}{a}$;

whence $v = \dfrac{r\alpha}{a}$, $\dfrac{dv}{du} = \dfrac{r}{a}\dfrac{d\alpha}{du^3}$; and for the whole of M,

$$\frac{\Sigma\,.\,mv}{u}\frac{dv}{du} = \frac{\Sigma\,.\,mr^2\alpha}{a^2u}\frac{d\alpha}{du} = \frac{\alpha}{a^2u}\frac{d\alpha}{du}\Sigma\,.\,mr^2$$

(by last Article) $= \dfrac{\alpha}{a^2u}\dfrac{d\alpha}{du}M(a^2+k^2) = M\dfrac{\alpha}{u}\dfrac{d\alpha}{du} + M\dfrac{k^2\alpha}{a^2u}\dfrac{d\alpha}{du}.$

Hence in the denominator of the accelerating force x, we shall have, for each mass M, two terms, such as we have just found: one of them depending on the quantity of matter M, and the other on the moment of inertia Mk^2.

If the machine be such that the velocities of the particles have to each other ratios constant for the same particle, let $a = nu$, where n does not vary with the time; therefore

$$\frac{da}{du} = n = \frac{a}{u}, \text{ and } \frac{a}{u}\,\frac{da}{du} = \frac{a^2}{u^2};$$

whence the two terms of the denominator of x, corresponding to the mass M, are

$$M\frac{a^2}{u^2} + M\frac{k^2a^2}{a^2u^2}.$$

Also $\frac{a}{a}$ is the angular velocity of M about its centre: therefore if ω be the angular velocity, the two terms corresponding to M will be

$$M\frac{a^2}{u^2} + M\frac{k^2\omega^2}{u^2}.$$

It will be seen by comparison, that this includes all the preceding propositions of this Chapter.

Sect. IV. MAXIMUM EFFECT OF MACHINES.

224. PROB. I. *A weight* P, *acting at a wheel, produces rotation in a mass* M, *which moves, with the wheel, about an axis passing through the centre of gravity; it is required to determine the distance at which* P *must act, that the angular velocity, generated in a given time, may be the greatest possible.*

Here the accelerating force on P is

$$f = \frac{Pa^2g}{Pa^2 + Mk^2},$$

P acting at a radius a. And the velocity generated in time t in the circumference at which P acts, is ft. And hence,

$$\text{angular velocity} = \frac{ft}{a}; \ \therefore \frac{f}{a} = \text{max.},$$

$$\frac{a}{Pa^2 + Mk^2} = \text{max.} \quad \frac{Pa^2 + Mk^2}{a} = \text{min.},$$

$$Pa + \frac{Mk^2}{a} = \text{min. whence } P - \frac{Mk^2}{a^2} = 0,$$

$$a = k\sqrt{\frac{M}{P}}.$$

PROB. II. P *raises* Q *by means of a wheel and axle, as in* Art. 213; *the axle being given, to find the wheel, that the time of* Q's *ascending through a given space, may be the least possible.*

The accelerating force on Q is

$$f = \frac{(Pa - Qb)gb}{Pa^2 + Qb^2 + Mk^2}.$$

And, as this is constant, we have $t = \sqrt{\frac{2s}{f}}$, which will manifestly be least when f is greatest. Therefore, we must have

$$\frac{Pa - Qb}{Pa^2 + Qb^2 + Mk^2} = \text{max.}$$

If we suppose a to vary, k will also vary in a manner depending on the form of the wheel; but if we suppose M to be small, we have, neglecting it,

$$\frac{Pa - Qb}{Pa^2 + Qb^2} = \text{max.}$$

and differentiating, supposing a variable,

$$P(Pa^2 + Qb^2) - 2Pa(Pa - Qb) = 0;$$

$$Pa^2 - 2Qab - Qb^2 = 0;$$

$$\therefore a = \frac{Qb}{P}\left\{1 + \sqrt{\left(1 + \frac{P}{Q}\right)}\right\}.$$

If P be small compared with Q, this will give nearly

$$a = \frac{2Qb}{P} + \frac{b}{2}.$$

The weight P must act at a little more than twice the distance at which it would balance Q.

Prob. III. *The wheel and axle, and the weight* P, *being given; to find* Q, *so that the momentum communicated to it in a given time, may be the greatest possible.*

The accelerating force f on Q, is the same as before, and the velocity acquired by it in a given time, is $v = ft$. And hence, we must have Qft a maximum, or Qf a maximum;

$$\therefore \quad \frac{(Pa - Qb)\,Q}{Pa^2 + Qb^2 + Mk^2} = \text{max.};$$

$$\therefore \;(Pa - 2Qb)\,(Pa^2 + Qb^2 + Mk^2) - b^2\,(PQa - Q^2b) = 0.$$

And hence, Q is found by the solution of a quadratic equation.

If we neglect M, we have

$$Q = \frac{Pa^2}{b^2}\left\{\sqrt{\left(1 + \frac{b}{a}\right)} - 1\right\};$$

and if we suppose b small, compared with a,

$$Q = \frac{Pa}{2b} - \frac{P}{8};$$

Q is nearly half the weight which P would support.

Prob. IV. *A body revolving round an axis, strikes another given body* P *in a direction perpendicular to the radius; to find the distance at which the impact must take place that the velocity communicated may be the greatest possible.*

Let r be the distance of the striking point from the axis, and Mk^2 the moment of inertia with respect to the axis. Any pressure acting at the distance r, will produce the same effect as if there were collected at that distance a mass $\frac{Mk^2}{r^2}$. Hence, the impact, which is only a short and violent pressure, will produce the same effect, as if such a mass were to impinge on the given body P. Let a be the angular velocity, then ra is the velocity at the point of impact. And when a body A impinges with a velocity a on B at rest, the velocity after impact is $\frac{Aa}{A+B}$; supposing the bodies inelastic, (*Elem. Treat. on Mechanics*, Art. 137.). Hence, the velocity communicated to P is

$$\frac{\frac{Mk^2a}{r}}{\frac{Mk^2}{r^2}+P} = \frac{Mk^2ar}{Mk^2+Pr^2},$$

which is to be a maximum. Whence we find

$$r = k\sqrt{\frac{M}{P}}.$$

If the body be in any degree elastic, the result will be the same, for the velocity of B in that case, is $\frac{(1+e)Aa}{A+B}$, (*Elem. Treat. on Mechanics*, Art. 138.).

CHAP. IV.

MOMENT OF INERTIA.

225. In the present Chapter we shall find the moment of inertia of a variety of different bodies, and with respect to any axis. From this it will be easy to deduce also the position of the centre of oscillation.

Sect. I. GENERAL FORMULÆ.

Since the expression for the moment of inertia of a system, consisting of a finite number of points, is $\Sigma(m\,.\,Cm^2)$, m being the portion of the mass which is at the distance Cm from the axis of rotation; when the number of points becomes indefinite, the expression will evidently become $\int_r \frac{dM}{dr} Cm^2$, where $\frac{dM}{dr}$ is the differential coefficient of the mass at the distance Cm. For we approach the true value by dividing the mass into portions smaller and smaller indefinitely, and taking the sum of each multiplied into the square of its distance; but by the nature of the differential calculus, we thus approach the differential of the mass at each point, multiplied into the square of its distance, and the integral of this. Hence, if r be the distance of any point from the axis, and $\frac{dM}{dr}$ the differential coefficient of the mass corresponding to r,

$$\text{moment of inertia} = \int r^2 \frac{dM}{dr} *.$$

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[*] This may be proved more distinctly thus:

Let AB, fig. 83, be any body, and $Cm = r$, $Cm' = r'$, be two distances from the axis: M the mass included within the distance r from the axis, M' that included within

This moment will be the same as that of the whole mass, collected at a certain distance. If we call this distance k, we have

$$k^2 M = \int_r r^2 \frac{dM}{dr}\text{*}.$$

We proceed to find $k^2 M$ in different bodies. We may consider bodies as being either physical lines, surfaces, or solids, and apply our formulæ to each. The moment itself will depend upon the thickness of the lines and surfaces, and the density of the substance; but if the mass be homogeneous, the line k will depend only upon the geometrical form of the system. The other quantities will enter as multipliers on both sides of the equation

$$k^2 M = \int_r r^2 \frac{dM}{dr}.$$

Hence, in finding k we may suppose the thickness of lines and surfaces, and the density of the mass, each to be unity.

The moment of inertia about an axis passing through the centre of gravity being found, the moment of inertia about any other axis parallel to this at the distance a, is $M(a^2 + k^2)$ by

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within the distance r'. And let the moment of inertia of the former portion $= R$, and of the latter $= R'$. Therefore, the moment of the portion mnm', included between the two distances, will be $R' - R$. And the portion itself will be $M' - M$.

Now $R' - R$ is evidently greater than $(M' - M) r^2$,

and less than $(M' - M) r'^2$;

$$\therefore \frac{R' - R}{M' - M} \text{ is} > r^2, \text{ and} < r'^2.$$

But, as M' and M become ultimately equal, r^2 and r'^2 become ultimately equal, and therefore $\frac{R' - R}{M' - M}$ is ultimately equal to either of them. Also, ultimately

$$\frac{R' - R}{M' - M} \text{ is } \frac{dR}{dM};$$

therefore, we have

$$\frac{dR}{dM} = r^2, \quad \frac{dR}{dr} = r^2 \frac{dM}{dr}, \quad R = \int_r r^2 \frac{dM}{dr}.$$

* The *centre of gyration* is defined to be 'the point at which the whole mass must be collected, that the rotatory motion communicated by a given force may be the same as before.' It appears by Chapter II, that *any point*, whose distance from the axis is k, possesses this property. We have called k the *radius of gyration*.

Art. 222. Hence, knowing the moment about the axis through the centre of gravity, we can find the moment about the other axis, and *vice versâ*.

Sect. II. MOMENT OF INERTIA OF A LINE, REVOLVING IN ITS OWN PLANE.

226. If any distance from the axis be r, and $\frac{ds}{dr}$ the differential coefficient of the length of the line, it appears by what has just been said, that we may suppose the thickness and density of the line each $=1$, and $\frac{dM}{dr}=\frac{ds}{dr}$. Then $k^2M=\int_r r^2\frac{ds}{dr}$.

We may take, for the centre of revolution, any point with respect to which the relation of $\frac{ds}{dr}$ and r is most simple; and then by Art. 222 find the moment of inertia about the centre of gravity, and from that, about any other point.

Ex. 1. To find k for a straight line revolving about an axis perpendicular to it in its middle point, fig. 56.

Call any distance $Cm=r$, $CA=CB=a$, $M=2a$,

$$2ak^2=\int_r r^2=\frac{r^3}{3}+\text{constant}.$$

and, the integral being taken from $r=-a$ to $r=a$,

$$2ak^2=\frac{2a^3}{3};\quad\therefore\ k^2=\frac{a^2}{3}.$$

Hence, momentum of inertia $=k^2M=\frac{a^2M}{3}$, when M is the mass of the line, and its thickness and density may be any whatever. (The thickness must necessarily be small, that it may be considered as a line).

Ex. 2. A straight line about an axis perpendicular to it through any point of it; (D, fig. 56.).

Let mass $= M$, $AB = 2a$, $CD = b$.

By Art. 222, mom. ab. $D =$ mom. ab. $C +$ mom. of M at dist. CD,

$$k^2 M = \frac{a^2 M}{3} + b^2 M,$$

$$k^2 = \frac{a^2}{3} + b^2.$$

Ex. 3. A straight line about any axis perpendicular to the plane in which it is.

Moment of AB, fig. 57, about E,

$$AB = 2a, \quad CD = b, \quad DE = c; \quad EC^2 = b^2 + c^2.$$

As before, $k^2 M =$ mom. round $C +$ mom. of M at dist. EC.

$$= \frac{a^2 M}{3} + (b^2 + c^2) M;$$

$$\therefore \; k^2 = \frac{a^2}{3} + b^2 + c^2.$$

Ex. 4. A circular arc about the centre of the circle.

If a be the radius, r always $= a$, and mom. of inertia $= a^2 M$;

$$\therefore \; k^2 = a^2.$$

Ex. 5. A circular arc about an axis perpendicular to its plane through its centre of gravity, fig. 58.

Let C be the centre of the arc PAp, G its centre of gravity. Then by the rules for the centre of gravity, (*Analytical Statics*, Art. 38. Ex. 21.)

$$CG = \frac{CA \,.\, \text{chord } Pp}{\text{arc } Pp} = \frac{aq}{p};$$

a being the radius, p the arc, q its chord.

Then, by Art. 220, mom. round $C =$ mom. round $G + CG^2 . M$;

$$\therefore \; \text{mom. round } G = \text{mom. round } C - CG^2 . M,$$

$$k^2 M = a^2 M - \frac{a^2 q}{p^2} M,$$

$$k^2 = a^2 \left(1 - \frac{q^2}{p^2}\right).$$

If APp be a semi-circle, $q = 2a$, $p = \pi a$,

$$k^2 = a^2 \left(1 - \frac{4}{\pi^2}\right).$$

Ex. 6. A circular arc about an axis perpendicular to its plane through its vertex A, fig. 58.

By the two last examples we shall here easily find

$$k^2 = 2a^2 \left(1 - \frac{q}{p}\right).$$

If the arc be a whole circumference, $q = 0$, $k^2 = 2a^2$.

Sect. III. MOMENT OF INERTIA OF A LINE, REVOLVING PERPENDICULARLY TO ITS OWN PLANE.

227. In such cases the figure revolves about a line in its own plane, or parallel to it. If this line divide the figure symmetrically, as AC, fig. 59, we shall have $\frac{dM}{dr} = 2 \times$ differential coefficient of the arc AQ, of which $CQ = r$, is the radius vector:

$$\text{and if } AQ = s, \quad k^2 M = 2 \textstyle\int_r r^2 \frac{ds}{dr}.$$

Ex. 7. A circular arc about a radius through its vertex A, fig. 59.

$$\text{Let } CA = a, \quad \frac{ds}{dr} = \frac{a}{\sqrt{(a^2 - r^2)}},$$

$$k^2 . \text{arc } PAp = \int_r \frac{2ar^2}{\sqrt{(a^2 - r^2)}}$$

$$= C - ar\sqrt{(a^2 - r^2)} + a^2 \text{ arc } \left(\sin = \frac{r}{a}\right).$$

And this, being taken to begin when $r = 0$, gives, puttting p for the arc PAp, and c for PM,

$$k^2 p = \frac{a^2}{2} p - ac\sqrt{(a^2 - c^2)},$$

$$k^2 = \frac{a^2}{2} - \frac{ac\sqrt{(a^2 - c^2)}}{p}.$$

If the arc be a semi-circle, $c = a$, $k^2 = \frac{a^2}{2}$.

The same is true if the arc be a circle, and $\therefore\ c = 0$.

Sect. IV. MOMENT OF INERTIA OF A SURFACE, REVOLVING IN ITS OWN PLANE.

228. If the mass be a plane surface defined by an equation between a radius vector r, and an angle θ, revolving about an axis perpendicular to the surface and passing through the origin, we shall have $\frac{dM}{dr} = \int_\theta r$, taken between proper limits of θ; $k^2 M = \int_r \int_\theta r^3$.

The integrations may be performed in any order. If we integrate first for r we shall have

$$k^2 M = \int_\theta \frac{r^4}{4};$$

the value of r being taken which belongs to the boundary of the figure. The same expression might also be obtained by conceiving the figure divided into triangles of indefinitely small width by lines drawn in it from the axis.

Ex. 8. To find the moment of inertia in any triangle, about an axis perpendicular to its plane, and through one of its angles.

Let ACB, fig. 60, be the triangle, and C its axis; CD perpendicular on $AB = h$. $CB = a$, $CA = b$, $AB = c$; $DCM = \theta$, $CM = r$,

$$\theta = \text{angle}\left(\cos. = \frac{h}{r}\right); \quad \frac{d\theta}{dr} = \frac{h}{r\sqrt{(r^2 - k)}},$$

$$k^2 M = \int_r \frac{hr^3}{4\sqrt{(r^2 - h^2)}} = \frac{hr^2 + 2h^3}{12}\sqrt{(r^2 - h^2)} + \text{constant},$$

and for the triangle ADC, we must take the integral from $r = h$ to $r = b$, which gives $\frac{b^2 + 2h^2}{12} h\sqrt{(b^2 - h^2)}$.

Similarly for BDC we have $\frac{a^2 + 2h^2}{12} h\sqrt{(a^2 - h^2)}$.

Hence, for ACB we have

$$k^2 M = \frac{a^2 + 2h^2}{12} h\sqrt{(a^2 - h^2)} + \frac{b^2 + 2h^2}{12} h\sqrt{(b^2 - h^2)}.$$

Cor. 1. If the triangle be isosceles, $k^2 M = \frac{a^2 + 2h^2}{6} . M$.

Cor. 2. Let AB be bisected in E, and let $DE = q$;

$$\therefore \; BD = \sqrt{(a^2 - h^2)} = \frac{c}{2} + q, \quad AD = \sqrt{(b^2 - h^2)} = \frac{c}{2} - q.$$

$$\text{And } BC^2 - BD^2 = AG^2 - AD^2,$$

$$\text{or } a^2 - \left(\frac{c}{2} + q\right)^2 = b^2 - \left(\frac{c}{2} - q\right)^2;$$

$$\therefore \; a^2 - b^2 = 2cq.$$

$$\text{Also } a^2 + 2h^2 = a^2 + 2\left\{a^2 - \left(\frac{c}{2} + q\right)^2\right\} = 3a^2 - 2\left(\frac{c}{2} + q\right)^2,$$

$$b^2 + 2h^2 = b^2 + 2\left\{b^2 - \left(\frac{c}{2} - q\right)^2\right\} = 3b^2 - 2\left(\frac{c}{2} + q\right)^2.$$

Hence,

$$k^2M = \frac{h}{12}\left\{3a^2\left(\frac{c}{2}+q\right) - 2\left(\frac{c}{2}+q\right)^3 + 3b^2\left(\frac{c}{2}-q\right) - 2\left(\frac{c}{2}-q\right)^3\right\}$$

$$= \frac{h}{12}\left\{\frac{3}{2}(a^2+b^2)c + 3(a^2-b^2)q - \frac{c^3}{2} - 6cq^2\right\},$$

or, since $3(a^2-b^2)q = 6cq^2$,

$$k^2M = \frac{h}{12}\left\{\frac{3}{2}(a^2+b^2)c - \frac{c^3}{2}\right\}$$

$$= \frac{ch}{24}\left\{3(a^2+b^2) - c^2\right\}$$

$$= \frac{M}{12}\left\{3(a^2+b^2) - c^2\right\};$$

$$\therefore\ k^2 = \frac{3(a^2+b^2) - c}{12}.$$

Ex. 9. A triangle about an axis through its centre of gravity.

By the property of the centre of gravity, (*Elem. Mech.* Chap. IV. Ex. 3.)

$$CG^2 = \frac{2(a^2+b^2) - c^2}{9};$$

$$\therefore\ \frac{M}{12}\left\{3(a^2+b^2) - c^2\right\} = M\,.\,\frac{2(a^2+b^2) - c^2}{9} + k^2M,$$

$$k^2M = M\left\{\frac{a^2+b^2+c^2}{36}\right\} = \frac{M(e^2+f^2+g^2)}{12};$$

e, f, g being the distances from G to the angles A, B, C.

If the triangle be equilateral,

$$k^2M = M\,.\,\frac{a^2}{12} = M\,.\,\frac{e^2}{4}.$$

Ex. 10. A parallelogram about an axis perpendicular to it through its centre of gravity.

Let $ABDE$, fig. 61, be the parallelogram, $AB = 2a$, $AD = 2b$, $BD = 2c$.

Let G be the centre of gravity of ABD; $\therefore$ by the *Statics*,

$$GC^2 = \frac{2(4a^2 + 4b^2) - 4c^2}{36} = \frac{2(a^2 + b^2) - c^2}{9}.$$

And mom. of ABD round C = mom. round $G + ABD \,.\, GC^2$

$$= ABD \left\{\frac{4a^2 + 4b^2 + 4c^2}{36} + \frac{2(a^2 + b^2) - c^2}{9}\right\}$$

$$= ABD \left\{\frac{a^2 + b^2}{3}\right\},$$

and doubling both sides, since $2ABD = M$,

$$k^2 M = M \,.\, \frac{a^2 + b^2}{3}, \text{ and } k^2 = \frac{a^2 + b^2}{3}.$$

Hence, it is independent of the angles of the parallelogram.

Ex. 11. Any regular polygon about an axis perpendicular to it through its centre.

The polygon may be divided into isosceles triangles, and for each of these, mom. $= \frac{a^2 + 2c^2}{6} A$, a and c being the radius of the circumscribed and inscribed circle, and A one of the triangles;

$$\therefore\ k^2 M = \frac{a^2 + 2c^2}{6} nA = \frac{a^2 + 2c^2}{6} M = \frac{a^2}{6}\left(1 + 2\cos.^2 \frac{\pi}{n}\right),$$

$$M = \frac{a^2}{6}\left(2 + \cos. \frac{2\pi}{n}\right) M,$$

$$\text{or, if } l \text{ be a side, } k^2 M = \frac{l^2}{24}\left\{\frac{1 + 2\cos.^2 \frac{\pi}{n}}{\sin.^2 \frac{\pi}{n}}\right\} M.$$

Ex. 12. A circle revolving about an axis perpendicular to it through its centre.

Here r is constant, and $= a$, the radius;

$$k^2 M = \int_\theta \frac{a^4}{4} = \frac{2\pi a^4}{4}, \text{ integrating for } \theta;$$

$$= \frac{\pi a^4}{2} = \frac{a^2 M}{2}, \quad k^2 = \frac{a^2}{2}.$$

Ex. 13. An annulus whose external and internal radii are a, b,

$$k^2 M = \frac{\pi(a^4 - b^4)}{2} = \frac{a^2 + b^2}{2} \cdot (\pi a^2 - \pi b^2) = \frac{a^2 + b^2}{2} M,$$

$$k^2 = \frac{a^2 + b^2}{2}.$$

Ex. 14. An ellipse revolving about the centre.

If θ be measured from the extremity of the major axis,

$$r^2 = \frac{b^2}{1 - e^2 \cos.^2 \theta};$$

$$\therefore\ k^2 M = b^4 \int_\theta \frac{1}{(2 - 2e^2 \cos.^2 \theta)^2}$$

$$= b^4 \int \frac{1}{(2 - e^2 - e^2 \cos. 2\theta)^2}$$

$$= \frac{b^4}{(2 - e^2)^2} \int \frac{1}{(1 - n \cos. 2\theta)^2},$$

putting n for $\frac{e^2}{2 - e^2}$.

To integrate, let $P = \frac{\sin. 2\theta}{1 - n \cos. 2\theta}$, and the integral $= S$,

$$\frac{dP}{d\theta} = \frac{2 \cos. 2\theta}{1 - n \cos. 2\theta} - \frac{2n \sin.^2 2\theta}{(1 - n \cos. 2\theta)^2}$$

$$= \frac{2\cos.2\theta - 2n}{(1 - n\cos.2\theta)^2}$$

$$= \frac{2}{n}\,\frac{(1 - n^2) - (1 - n\cos.2\theta)}{(1 - n\cos.2\theta)^2}$$

$$= \frac{2(1 - n^2)}{n}.\frac{1}{(1 - n\cos.2\theta)^2} - \frac{1}{n}\,\frac{2}{1 - n\cos.2\theta};$$

$$\therefore\ P = \frac{2(1 - n^2)}{n}.S - \frac{1}{n}\int_\theta \frac{1}{1 - n\cos.2\theta}.$$

And this latter integral is (*Lacroix*, Elem. Treat. Note K),

$$\frac{1}{\sqrt{(1 - n^2)}}\,\text{arc}\left(\cos. = \frac{\cos.2\theta - n}{1 - n\cos.2\theta}\right) = \frac{1}{\sqrt{(1 - n^2)}}A: \text{ suppose,}$$

$$\therefore\ S = \frac{1}{2(1 - n^2)}.P + \frac{1}{2(1 - n^2)^{\frac{3}{2}}}A.$$

If we take the integrals from $\theta = 0$, to $\theta = \pi$, and then double them, in order to obtain the whole value of S, we have

$$S = \frac{1}{(1 - n^2)^{\frac{3}{2}}}\,\text{arc}\,(\cos. = 1)$$

$$= \frac{2\pi}{(1 - n^2)^{\frac{3}{2}}};$$

$$\therefore\ k^2 M = \frac{2\pi b^4}{(2 - e^2)^2 (1 - n^2)^{\frac{3}{2}}};$$

and putting $a\sqrt{(1 - e^2)}$ for b, and $\frac{e}{\sqrt{(2 - e^2)}}$ for n,

$$k^2 M = \frac{\pi a^4}{4}(2 - e^2)\sqrt{(1 - e^2)}.$$

But the area $M = \pi a^2\sqrt{(1 - e^2)}$;

$$\therefore\ k^2 = \frac{a^2(2 - e^2)}{4} = \frac{a^2 + a^2(1 - e^2)}{4} = \frac{a^2 + b^2}{4}.$$

When $e = 0$, $k^2 = \frac{a^2}{2}$, as in a circle.

When $e = 1$, $b = 0$, $k^2 = \frac{a^2}{4}$.

229. If the surface be bounded by rectangular co-ordinates, we may thus find its moment when revolving in its own plane.

Let the centre of motion C, fig. 62, be the origin of co-ordinates $CN = x$, $NQ = y$. And we have

moment of Qq about C = moment of Qq about $N + CN^2 . Qq$, (by Ex. 1, Art. 79.),

$$= \frac{y^2}{3} . 2y + x^2 . 2y.$$

And integrating with respect to x,

$$k^2 M = 2 \int_x \left(\frac{y^3}{3} + x^2 y\right).$$

Ex. 15. A parabola revolving about its vertex. Here $y = a^{\frac{1}{2}} x^{\frac{1}{2}}$,

$$k^2 M = 2 \int_x \left(\frac{a^{\frac{3}{2}} x^{\frac{3}{2}}}{3} + a^{\frac{1}{2}} x^{\frac{5}{2}}\right) = 2 \left(\frac{2a^{\frac{3}{2}} x^{\frac{5}{2}}}{15} + \frac{2a^{\frac{1}{2}} x^{\frac{7}{2}}}{7}\right),$$

$$\text{and } M = \frac{4}{3} a^{\frac{1}{2}} x^{\frac{3}{2}}; \therefore k^2 = \frac{3x^2}{7} + \frac{y^2}{5},$$

x and y being the extreme ordinates.

Sect. V. MOMENT OF INERTIA OF A PLANE REVOLVING ABOUT AN AXIS IN OR PARALLEL TO THE PLANE.

230. When any plane revolves about a line *in it*, we may call r the distance of any point from the axis; and if $\frac{dz}{dr}$ be the

differential coefficient of the axis, $\int_x \frac{dx}{dr}$ integrated with respect to x, will be the differential coefficient with respect to r, and $\int_r \int_x r^2 \frac{dx}{dr}$ the moment of inertia.

The integral $\int_r \int_x r^2 \frac{dx}{dr}$ is $= \int_x \frac{r^3}{3}$; and r is given in terms of x.

Ex. 16. A circle revolving about a diameter.

Let radius $= a$, x measured from the vertex;

$$\therefore\ a - x = \sqrt{(a^2 - r^2)},$$

$$\frac{dx}{dr} = \frac{r}{\sqrt{(a^2 - r^2)}};\quad k^2 M = \frac{2}{3}\int_r \frac{r^4}{\sqrt{(a^2 - r^2)}},$$

which, integrated from $x = 0$, to $x = a$, gives for a semi-circle,

$$k^2 M = \frac{\pi a^4}{8}.$$

And since $M = \frac{\pi a^2}{2}$, $k^2 = \frac{a^2}{4}$.

The same expression is true for a whole circle.

Also for an ellipse revolving about either principal axis, $2a$ being the other.

Ex. 17. A circle revolving about a line parallel to its plane, at a distance c from its centre; radius $= a$.

A diameter being drawn parallel to the axis of rotation, we have, by last Example,

$$\text{moment round diameter} = \frac{a^2}{4} M.$$

And therefore by Art. 222,

$$k^2 M = \frac{a^2}{4} M + c^2 M,$$

$$k^2 = \frac{a^2}{4} + c^2.$$

Ex. 18. **An isosceles triangle about its perpendicular.**

Perpendicular $= a$, base $= 2b$; x measured from vertex,

$$r = \frac{bx}{a}, \quad k^2 M = \frac{2b^3}{3a^3} \int_x x^3 = \frac{b^3 x^4}{6a^3} = \frac{b^3 a}{6}.$$

But $M = ba$; $\therefore\ k^2 = \frac{b^2}{6}$.

Sect. VI. Moment of Inertia of a symmetrical Solid about its Axis.

231. When we have a solid of which all the sections perpendicular to the axis are similar, it may, by planes perpendicular to this axis, be divided into indefinitely thin slices, and the moment of inertia of the whole, will be the sum of the moments of these parts; and ultimately, it will be the integral of a portion of the moment which corresponds to the differential coefficient with respect to the axis. This portion will be found by taking the moment of the plane, which is the section of the solid, (found by last Article).

Ex. 19. **A cone revolving about its axis.**

The momentum of a circle, radius $= r$, is $\frac{\pi r^4}{2}$. Also in the cone r is as x: let $r = nx$: then the differential coefficient of the cone with reference to x is $\frac{\pi r^4}{2} = \frac{\pi n^4 x^4}{2}$,

$$k^2 M = \frac{\pi n^4 x^5}{10}, \text{ and taken to } x = a,$$

$$= \frac{\pi n^4 a^5}{10} = \frac{\pi b^4 a}{10},$$

if $na = b$, the radius of the base.

$$\text{And } M = \frac{\pi b^2 a}{3}; \quad \therefore\ k^2 = \frac{3b^2}{10}.$$

Ex. 20. A sphere about a diameter,

$$k^2 = \frac{2a^2}{5}.$$

Ex. 21. A hollow sphere, of which the external and internal radii are a, b,

$$k^2 = \frac{2}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3}.$$

Ex. 22. An ellipsoid about its axis: the semi-axes of the largest section perpendicular to the axis of rotation being a, b,

$$k^2 = \frac{a^2 + b^2}{5}.$$

Ex. 23. A parallelepiped, $k^2 = \frac{a^2 + b^2}{3}$.

Ex. 24. A cylinder, $k^2 = \frac{a^2}{2}$.

Ex. 25. A hollow cylinder, $k^2 = \frac{a^2 + b^2}{2}$.

Sect. VII. MOMENT OF INERTIA OF A SOLID NOT SYMMETRICAL.

232. When the solid comes under this description, different methods may be used, as in the following Examples.

Ex. 26. A cylinder about an axis perpendicular to its own, through its middle.

Let its radius $= b$, its length $= 2a$.

A circle perpendicular to the axis of the cylinder, and at a distance x from the axis of rotation, has its moment, (Ex. 17.)

$$= \pi b^2 \left(x^2 + \frac{b^2}{4}\right);$$

$$\therefore\ k^2 M = \int_x \pi b^2 \left\{x^2 + \frac{b^2}{4}\right\} = \pi b^2 \left\{\frac{x^3}{3} + \frac{b^2 x}{4}\right\}$$

$$= \pi b^2 \left\{\frac{2a^3}{3} + \frac{b^2 a}{2}\right\} = 2\pi b^2 a \left\{\frac{a^2}{3} + \frac{b^2}{4}\right\},$$

and since $M = 2\pi b^2 a$, $k^2 = \dfrac{a^2}{3} + \dfrac{b^2}{4}$.

Ex. 27. A cone about an axis perpendicular to its axis at its vertex.

Fig. 63. AB the base, CD the axis of rotation, MP any section parallel to the base,

$$CM = x, \quad MP = nx, \quad CA = a, \quad AB = b = na.$$

By Ex. 17, moment of circle MP round CD

$$= \text{circle}\,.\left(x^2 + \frac{n^2 x^2}{4}\right)$$

$$= \pi n^2 \left(1 + \frac{n^2}{4}\right) x^4;$$

$$\therefore\ k^2 M = \pi n^2 \left(1 + \frac{n^2}{4}\right) \int_x x^4 = \frac{\pi n^2 x^5}{5} \left(1 + \frac{n^2}{4}\right)$$

$$= \pi n^2 a^5 \left(1 + \frac{n^2}{4}\right),$$

and $M = \dfrac{\pi n^2 a^2 . a}{3}$; $\therefore\ k^2 = \dfrac{3a^2}{5}\left(1 + \dfrac{n^2}{4}\right) = \dfrac{3}{5}\left(a^2 + \dfrac{b^2}{4}\right)$.

Ex. 28. A cone about an axis perpendicular to the axis of the cone, and passing through its centre of gravity.

The distance of the centre of gravity from the vertex is $\frac{3a}{4}$; hence, by Art. 222,

$$\text{moment round } CD = \text{moment round } GH + CG^2 . M,$$

$$\frac{3}{5}\left(a^2 + \frac{b^2}{4}\right) M = k^2 M + \frac{9a^2}{16} M,$$

$$k^2 = \frac{3(a^2 + 4b^2)}{80}.$$

And similarly for other figures of revolution*.

Sect. VIII. CENTRE OF OSCILLATION.

233. Prop. *To find the momentum of inertia of a given body, by a practical method.*

Let the body AB, fig. 64, be suspended from an horizontal axis C; and let a vertical plane be drawn through the axis, which will be the plane in which the centre of gravity is. Let the body be moved, so that this plane may be exactly horizontal (CB), and let the weight Q be ascertained, which, acting vertically at B, will support the body in this position. If W be the weight of the mass, we shall have

$$W = \frac{Q . CB}{CG}.$$

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* The expressions in this Chapter may also be found by using the following formulæ in rectangular co-ordinates. The body revolves about the axis of z.

For planes, $\int_x \int_y (x^2 + y^2)$;

where, after the first integration, in x for instance, limits are to be put in terms of y.

For solids, $\int_x \int_y \int_z (x^2 + y^2) z = \int_x \int_y (x^2 + y^2) z$;

where z must be put in terms of x and y, and then the integrations performed, and the limits introduced as before. See Poisson, *Traitè de Mec.* No. 348, 349.

Now let the body hang from C, and make small oscillations, and let it make n oscillations in a time t. Then the time of one oscillation is, by Cor. 1, Art. 193.

$$\frac{t}{n} = \pi\sqrt{\frac{CO}{g}};$$

O being the centre of oscillation;

$$\therefore\ CO = \frac{gt^2}{\pi^2 n^2}.\quad \text{And by Art. 208.}$$

$$\Sigma(m\,.\,Cm^2) = CG\,.\,CO\,.\,\Sigma m = \frac{Q}{W}\cdot\frac{CB\,.\,g\,.\,t^2}{\pi^2 n^2}\,.\,M,$$

M being the mass of the system;

$$\therefore\ k^2 = \frac{Q}{W}\cdot\frac{t^2}{\pi^2 n^2}\,.\,CB\,.\,g.$$

234. Prop. *The centres of suspension and oscillation are reciprocal.*

That is, if the centre of oscillation be made the point of suspension, the former point of suspension will become the centre of oscillation.

$$\text{In fig. 53,}\quad CO = \frac{\Sigma(m\,.\,Cm^2)}{CG\,.\,\Sigma m}.$$

But by Art. 222.

$$\Sigma(m\,.\,Cm^2) = \Sigma(m\,.\,Gm^2) + CG^2\,.\,\Sigma m;$$

or, putting M for Σm the mass, and $k^2 M$ for $\Sigma(m\,.\,Gm^2)$,

$$CO = \frac{k^2}{CG} + CG,\ \text{and}\ GO = \frac{k^2}{CG},\ \text{and}\ CG = \frac{k^2}{GO}.$$

If therefore, we suspend the body from O, C will be its centre of oscillation.

Cor. 1. CO depends on CG alone, and will be the same, so long as CG is the same. Hence, if with centre G, (fig. 65,) and radius *GC*, we describe a circle; (in the plane of oscillation, that is, a plane perpendicular to the axis of suspension;) *CO* is the same from whatever point of this circumference we suspend the body. And therefore, the time of oscillation is the same.

Cor. 2. Also, if we describe a circle with radius *GO*, the time of oscillation will be the same, whether the body is suspended from any point in the circumference *OO′*, or from any point in the circumference *CC′*.

235. Prop. *To find from what axis a body must be suspended, that it may oscillate in the least time possible;*

That is, amongst all the axes which are parallel to one another.

The time of oscillation is that of a simple pendulum, whose length is *CO*, fig. 65, *C* being the axis of suspension, *O* the centre of oscillation. Therefore, the time will be least, when *CO* is least;

$$\therefore \text{ by last Article, } CG + \frac{k^2}{CG} = \text{minimum};$$

and taking the differential coefficient,

$$1 - \frac{k^2}{CG^2} = 0; \quad \therefore k = CG.$$

Hence, the time will be the least when $CG = k$.

$$\text{In that case } GO = \frac{k^2}{CG} = k \text{ also.}$$

Hence, the least time of oscillation will take place when the body is suspended from *K*, so that *L* being the centre of oscillation, $KG = GL$.

Cor. 1. The least time of oscillation is equal to that of a simple pendulum, whose length is $2k$.

Cor. 2. If we describe a circle with radius *GK*, the time will be the same when the body is suspended from any point in this circumference *KK′*.

Cor. 3. It has been said in last Article, that the times of oscillation (t), are the same when the point of suspension is in any point of the circumferences CC', OO'; for a point between these circumferences, the time is less than t; and it is least for a point in the circumference KK'. For points within OO', or without CC', the time is greater than t, and becomes infinite as the point of suspension approaches the centre, or goes off to an infinite distance.

236. Prop. *To find the centre of oscillation in given figures.*

We have, (Art. 208.)

$$CO = \frac{\text{moment of inertia about axis}}{\text{mass} \,.\, CG} \,\ldots\ldots\ldots (a).$$

$$\text{Also, } GO = \frac{k^2}{CG} \ldots\ldots\ldots (a');$$

$k^2 M$ being the moment of inertia about an axis through the centre of gravity, and parallel to the axis of suspension. And either of these formulæ, by the assistance of the preceding part of this Chapter, will enable us to find O.

Ex. 1. To find the centre of oscillation of a straight line, suspended from its extremity.

By Ex. 1, Art. 226, $k^2 = \frac{a^2}{3}$, $2a$ being the length of the line;

$\therefore$ by (a'), $GO = \frac{a}{3}$ for $CG = a$. And $CO = a + \frac{a}{3} = \frac{2}{3} \,.\, 2a$.

Ex. 2. A line AB, fig. 66, oscillating in its own plane.

G the middle point, $CG = l$, $AB = 2a$;

$$GO = \frac{k^2}{l} = \frac{a^2}{3l};$$

$\therefore$ GO is independent of the angle COA.

Ex. 3. A circle *AB*, fig. 67, oscillating in its own plane.

$CG = l$, radius a; and by Ex. 12, Art. 228, $k^2 = \frac{a^2}{2}$;

$$\therefore\ GO = \frac{k^2}{l} = \frac{a^2}{2l}.$$

If C be in the circumference,

$$GO = \frac{a}{2},\ CO = \frac{3a}{2}.$$

The same is true for a cylinder, about a line parallel to its axis.

Ex. 4. A circle oscillating about an axis in its own plane; (about *CD*, fig. 67.)

By Ex. 16, Art. 230,

$$k^2 = \frac{a^2}{4};\ GO = \frac{a^2}{4l}.$$

If the axis be a tangent to the circumference,

$$l = a,\ GO = \frac{a}{4},\ CO = \frac{5a}{4}.$$

Ex. 5. A globe about any axis.

Distance to the centre of the globe $= l$, radius $= a$.

By Ex. 20, Art. 231,

$$k^2 = \frac{2a^2}{5},\ GO = \frac{2a^2}{5l},\ CO = l + \frac{2a^2}{5l}.$$

Ex. 6. A cone about its vertex.

Axis $= a$, radius of base $= b$.

By Ex. 27, Art. 232,

moment of inertia $= \frac{3}{5}\left(a^2 + \frac{b^2}{4}\right) M$. Also $CG = \frac{3a}{4}$;

$$\therefore\ CO = \frac{4}{5}\left(a + \frac{b^2}{4a}\right) = \frac{4a}{5} + \frac{b^2}{5a}.$$

That the centre of oscillation may be in the centre of the base, we must have $CO=a$; whence $b=a$, and the cone is a right-angled one.

Ex. 7. Let a solid, composed of two equal cones set base to base, oscillate in the direction of its axis.

Let G, fig. 68, be the common centre of the bases, GB a horizontal radius, and therefore parallel to CD the axis of rotation.

We must find the moment of inertia of the figure about GB. Now for the cone GA, we have, if F be its centre of gravity, and FK parallel to GB,

$$\text{moment about } GB = \text{moment about } FK + GF^2 \,.\, \text{cone},$$

$$\text{by Ex. 28,} \quad = \frac{3\,(b^2+4b^2)}{80} \,.\, \text{cone} + \frac{a^2}{16} \,.\, \text{cone}$$

$$= \frac{2a^2+3b^2}{20} \,.\, \text{cone}.$$

And hence the whole moment, or

$$k^2 M = \frac{2a^2+3b^2}{20} \,.\, 2 \,.\, \text{cone};$$

$$\therefore\ k^2 = \frac{2a^2+3b^2}{20}. \quad \text{And if } CG = l,$$

$$CO = l + \frac{k^2}{l} = l + \frac{2a^2+3b^2}{20l}.$$

237. Prop. *Having a system composed of several separate bodies, whose centres of gravity and oscillation are known; to find the centre of oscillation of the whole.*

In fig. 69, let g, o, be the centres of gravity and oscillation of m; g', o', those of m'; g'', o'', those of m''; and so on. And let G be the centre of gravity, and O that of oscillation for the whole system. Also let gh, $g'h'$, $g''h''$, &c. be perpendiculars on CG.

$$\text{moment of inertia of } m \text{ about } C = m \,.\, Cg \,.\, Co,$$

$$\text{of } m' \ldots\ldots\ldots = m' \,.\, Cg' \,.\, Co',$$

$$\text{\&c.} \ldots\ldots\ldots\ldots = \text{\&c.}$$

$$\therefore \text{ whole moment} = m \,.\, Cg \,.\, Co + m' \,.\, Cg' \,.\, Co' + \text{\&c.}$$

$$\therefore \; CO = \frac{m \,.\, Cg \,.\, Co + m' \,.\, Cg' \,.\, Co' + \text{\&c.}}{(m + m' + \text{\&c.})\, CG}$$

$$= \frac{m \,.\, Cg \,.\, Co + m' \,.\, Cg' \,.\, Co' + \text{\&c.}}{m \,.\, Ch + m' \,.\, Ch' + \text{\&c.}}.$$

238. PROP. *To find practically the length of a pendulum, which oscillates seconds.*

If we know the exact length of a simple pendulum which makes a given number of small oscillations in 24 hours, we can find the length of a pendulum which will oscillate in any given time, as 1 second. But it is impossible to form a pendulum which may, with sufficient regard for accuracy, be considered as a simple pendulum, that is, as a single point suspended by a string without weight. It is necessary, therefore, in our experiment, to find the distance between the centre of suspension and the centre of oscillation, of the oscillating body: and the difficulty of the case is to determine accurately this latter point; for the unavoidable irregularities of figure and density make its geometrical determination include errors which the delicacy of the inquiry renders important.

To avoid the sources of inaccuracy, Captain Kater has ingeniously employed the property of a compound pendulum, proved in Art. 234; viz., that the centres of oscillation and suspension are reciprocal. It follows from that property, that if a pendulum have two centres of suspension, and oscillate on them successively, first with one end uppermost, and then with the other, so that the times of oscillation in the two cases may be exactly equal; the distance of these two centres will be the length of the equivalent simple pendulum, whatever be the irregularities of form or composition in the instrument. The manner in which this result was produced, was as follows:

A brass pendulum CD, fig. 70, was furnished with two axes, from which it could be suspended; one passing through C, and the other through O. Besides the principal weight D, it was provided with a smaller sliding weight F, which could be moved along the stem CD; and this weight was to be moved till the number of oscillations in a given time, (as 24 hours,) was the same, whether the pendulum was suspended from C or from O.

F was placed in such a position, that by moving it towards C, as to f, the number of oscillations about C in twenty four hours was increased; and by the same change, the number of oscillations about O in the same time was also increased, but not so much. We shall afterwards consider this position mathematically. The adjustment was thus made.

Let the weight be at F, and let the number of oscillations in 10^{m} about O be 606, and about C be 601. Now let F be moved to f; and let the oscillations in 10^{m} be 607 about O, and 609 about C, (because the latter are more affected than the former). Then, the proper position of the slider is somewhere between F and f. Let it be placed at f', bisecting Ff; and let the oscillations in this case about O and C be $606\frac{1}{2}$, and 606; then, the proper position is between f and f'; and so on. Observing always, that if the number of vibrations about O be the greater, the slider must move toward C; and if the contrary, it must move towards O. By this means, continually halving the distance last moved, we may make the oscillations about C and O approach within any required degree of exactness. The distance of C and O being then measured, will give the length of a pendulum which makes a known number of oscillations in 10 minutes*.

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* There were, in Captain Kater's experiments, a number of contrivances which it would detain us too long to describe. Besides the slider F, he had another moveable weight E; and he made the numbers of oscillations nearly equal by means of this, before he attempted a more accurate adjustment by the slider.

The axes through C and O, were made with *knife edges*, which resting on planes of agate, turned as nearly as possible on a mathematical line. It is however true, as has been

239. **Prop.** *To determine the effect produced by the change of position of the moveable weight in last Article.*

We shall consider any pendulum, and a small weight moveable along the line passing through the centres of suspension and gravity.

Let m be the mass of the pendulum, independent of the moveable weight; h the distance of the centre of gravity of the pendulum from the centre of suspension C, l the distance of its centre of oscillation from C. And let μ be the moveable weight, λ the distance from C of the centre of gravity of μ, or of oscillation, supposing these centres to coincide because the weight μ is small. And let L be the length of the corresponding simple pendulum; then by Art. 237,

$$L = \frac{mhl + \mu\lambda^2}{mh + \mu\lambda}; \text{ and, } L \text{ and } \lambda \text{ being variable,}$$

$$\frac{dL}{d\lambda} = \frac{\mu^2\lambda^2 + 2m\mu h\lambda - m\mu hl}{(mh + \mu\lambda)^2}$$

$$= \frac{\mu^2 (\lambda - a)(\lambda + \beta)}{(mh + \mu\lambda)^2};$$

where $\mu a = \sqrt{(m^2h^2 + m\mu hl)} - mh$;

$$\mu\beta = \sqrt{(m^2h^2 + m\mu hl)} + mh.$$

When $\lambda = a$, $\dfrac{dL}{d\lambda} = 0$; hence if $Cx = a$, and if the weight F be placed at x, a small change in the position of this weight

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been proved by Laplace, and will be shewn hereafter, that, if they had been equal cylinders, their distance would still have given exactly the length of the simple pendulum. The method of determining the number of vibrations in 24 hours was ingenious; it was done by placing the pendulum in front of a clock pendulum, oscillating nearly in the same time; and observing the intervals at which the two pendulums coincided. Corrections were also to be made, for the magnitude of the arc vibrated; for the buoyancy of the atmosphere; for the temperature, &c. The reader will find Captain Kater's account of his method, and its results, in the *Phil. Trans.* for 1818, p. 33.

will not alter the time of oscillation about C. If F be below x, λ is greater than a, and $\frac{dL}{d\lambda}$ is positive; and therefore, if F be moved towards C, that is, if λ become less, L will also be less, and the time of oscillation about C will be shortened. If F be moved from C, the time will be lengthened. If F be above x, the contrary effects will follow.

In like manner, when the pendulum oscillates about O, if L' be the length of the corresponding simple pendulum, we shall have

$$\frac{dL'}{d\lambda'} = \frac{\mu^2 (\lambda' - a')(\lambda' + \beta')}{(mh' + \mu\lambda')^2},$$

$$\text{where } \mu a' = \sqrt{(m^2 h^2 + m\mu h' l')} - mh',$$

h', l', λ', a', β', being the quantities analogous to h, l, λ, a, β, but measured from O. And if $Oy = a'$, and if F be nearer to O than y is, we shall have $\lambda' < a'$, $\frac{dL'}{d\lambda'}$ will be negative, and an increase of λ' will produce a diminution of L'; that is, if F be moved towards C, the time of oscillation about O will be shortened.

Hence in this case, that is, when both the points x, y, are on the same side of F on which C is, if F be moved towards C the time of oscillation will be shortened, both about C and about O; but it will be shortened in different degrees in consequence of the different values of the two coefficients $\frac{dL}{d\lambda}$, $\frac{dL'}{d\lambda'}$,

When F is not far from one of the two points x, y, as for instance from y, the variation of the value of L' will depend principally upon the value of the factor $\lambda' - a'$, that is, Fy in the figure; and it is clear that except xy in the figure be very small, Fy and Fx will have very different values, and L and L' will vary in a very different degree for the same variation of λ or λ'.

240. Let the distance CO, of the two points of suspension be $2a$; and M being the middle point of CO, let F be the position of the moveable weight, when the oscillations about C and about O are performed in the same time; therefore in this case $L = CO = 2a$. Let $MF = \delta$; therefore, when $L = 2a$, $\lambda = a + \delta$. Hence,

$$2a = \frac{mhl + \mu(a+\delta)^2}{mh + \mu(a+\delta)};$$

$$\therefore\ l = 2a + \frac{\mu(a^2 - \delta^2)}{mh}.$$

In like manner, let the pendulum be suspended from O, and let l', h', be the quantities analogous to l, h; then

$$l' = 2a + \frac{\mu(a^2 - \delta^2)}{mh'}.$$

Now by expansion,

$$a = \tfrac{1}{2}l - \frac{\mu}{8m}\frac{l^2}{h} + \&c.$$

$$a' = \tfrac{1}{2}l' - \frac{\mu}{8m}\frac{l'^2}{h'} + \&c.$$

and substituting the values of l, neglecting terms involving $\left(\frac{\mu}{m}\right)^2$, since μ is small in comparison with m;

$$a = a + \frac{\mu(a^2 - \delta^2)}{2mh} - \frac{\mu \cdot 4a^2}{8mh}$$

$$= a - \frac{\mu\delta^2}{2mh}.$$

Similarly

$$a' = a - \frac{\mu\delta^2}{2mh'}.$$

Hence x is between M and C, and y is between M and O.

If the weight F be placed between x and y, $\lambda - a$ and $\lambda' - a'$ are both positive, whence $\frac{dL}{d\lambda}$ and $\frac{dL'}{d\lambda'}$, are both positive. Hence if F be moved towards C, L is shortened and L' is lengthened, and *vice versa*. In this case therefore, if it appear by the times of oscillation that L is greater than L', they may be made to approach to equality by moving F towards C; and in like manner if L' be greater than L, they may be made to approach to equality by moving F the other way.

But the times about C and O cannot be made equal, so long as F is between x and y; for

$$My = \frac{\mu\delta^2}{2mh} = \frac{\mu\delta}{2mh} MF;$$

and therefore My is less than MF, F being the requisite position of the moveable weight.

CHAP. V.

PRESSURE ON A FIXED AXIS.

241. When a body revolves about a fixed axis, the axis in general suffers some pressure, depending upon the form and motion of the body, and upon the forces which act upon the body. The different parts of the system influence each other's motions by the intervention of this axis; and it supplies, as it were, the difference of the *forces impressed*, and the *effective forces*, so that they may balance each other. The principle stated in Article 202, being general for *all* the forces which act upon a system, will enable us to find the pressures in question.

If a body, not acted upon by any forces, have a rotatory motion, it will retain it for ever, (neglecting friction, &c.) and will go on with a uniform velocity: if it be acted upon by external forces, its velocity will be variable. We shall consider successively these two cases.

Sect. I. A Body revolving, acted on by no Forces.

Prop. *A system acted on by no forces, revolves about a fixed axis with* uniform *velocity; to find the pressure on the axis.*

Let zC, fig. 83, be the axis of rotation, xCy a plane perpendicular to it, in which Cx and Cy are at right angles. The system may be referred to three rectangular co-ordinates, parallel to Cx, Cy, Cz, which we shall call x, y, z. Let M be any particle of the body, and MO perpendicular to the axis: M describes a circle about O, and for this purpose, it must be retained in a circle by a force in the direction MO, the magnitude of which force is known from Art. 27: if ω be the angular velocity, and $OM = r$, *the effective accelerating force in*

$MO = r\omega^2$. Hence, if m be the mass of the particle at M, $mr\omega^2$ will be the *effective moving force* in MO.

The force in MO may be resolved in the directions MN, NO, parallel to Cx and Cy. And we shall have for the particle m,

$$\text{moving force parallel to } x = mr\omega^2 \frac{MN}{MO} = mx\omega^2;$$

$$\text{to } y = mr\omega^2 \frac{NO}{MO} = my\omega^2:$$

and *effective forces*, analogous to these, act on each of the points of the system.

Also, the moments of these forces will be

about the axis Cx, 0, and $myz\omega^2$,
about the axis Cy, $mxz\omega^2$, and 0.

And if Σ represent the sum of all the products corresponding to different points analogous to m, we shall have for the *whole effective forces*, $\omega^2\Sigma . mx$, $\omega^2\Sigma . my$, and for their moments about Cx, and Cy, $\omega^2\Sigma . myz$, $\omega^2\Sigma . mxz$, these forces acting to diminish x and y.

The *impressed forces* are none, except the reaction of the axis, which is equal to the pressures upon it, taken in the opposite direction. We may reduce these pressures to two*, acting at given points.

If the axis be supported at two points A and B, we may suppose the forces to act at these points. Let U and V be the forces which act at those points, both being in planes perpendicular to the axis; for it is evident that there will be no pressure in the direction of the axis. Let U make an angle ϕ with the plane xz, and V an angle ψ with the same plane: and let $CA = a$, $CB = b$. Then the pressures on the axis will be

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* They cannot always be reduced to one, as will be seen.

When forces are reducible to two equal ones, acting on a line in opposite directions, at different points, the effect produced is of a peculiar kind, and may be called *Torsion*. There is no tendency to a change of *place*, but only of position. No forces so circumstanced are called a *couple*. The axis about which this torsion takes place, passes through the centre of gravity, as will be shewn.

$$\text{parallel to } x, \quad U \cos. \phi \text{ at } A, \quad V \cos. \psi \text{ at } B,$$

$$\text{parallel to } y, \quad U \sin. \phi \text{ at } A, \quad V \sin. \psi \text{ at } B.$$

Hence, by Article 202, $U \cos. \psi + V \cos. \psi = \omega^2 \Sigma mx$,

$$U \sin. \phi + V \sin. \psi = \omega^2 \Sigma my;$$

because the forces must be equal.

$$\text{And } \; Ua \cos. \phi - Vb \cos. \psi = \omega^2 \Sigma mxz,$$

$$Ua \sin. \phi - Vb \sin. \psi = \omega^2 \Sigma myx;$$

because the moments of the forces, with respect to Cx and Cy must be equal.

From these four equations, a and b being known, we may find the four quantities U, V, ϕ, ψ, (or rather, $U \cos. \phi$, $V \cos. \psi$, $U \sin. \phi$, $V \sin. \psi$.)

242. We may suppose C to be taken so that the plane xCy, fig. 84, passes through the centre of gravity G: and we shall be able to reduce all the forces to

(1) A force CR in the plane xCy.

(2) Two equal forces FS, $F'S'$ at equal distances CF, CF', and acting in opposite directions.

These latter forces will produce no effect, except a *torsion* round C. The tendency to a motion of the centre of gravity will entirely result from the force CR.

Prop. *In a revolving system as before, it is required to find the force which acts at the point* C *of the axis, and the forces which tend to turn the system round* C.

Let the former force be R, and make with the plane xz an angle ρ; let the forces at F and F' be each $\frac{1}{2} S$, and make with

xz an angle σ; and let $CF = CF' = f$. We shall then have impressed forces

$$\text{parallel to } x,\ R\cos.\rho \text{ at } C,\ \frac{S}{2}\cos.\sigma \text{ at } F,\ -\frac{S}{2}\cos.\sigma \text{ at } F',$$

$$\text{parallel to } y,\ R\sin.\rho \text{ at } C,\ \frac{S}{2}\sin.\sigma \text{ at } F,\ -\frac{S}{2}\sin.\sigma \text{ at } F'.$$

And considering the moments of the forces with respect to the axes Cx and Cy, we have, as before, by the conditions of equilibrium with the effective forces,

$$R\cos.\rho = \omega^2\Sigma mx,\quad R\sin.\rho = \omega^2\Sigma my,$$

$$Sf\cos.\sigma = \omega^2\Sigma mxz,\quad Sf\sin.\sigma = \omega^2\Sigma myz.$$

The two former equations give

$$\tan.\rho = \frac{\Sigma my}{\Sigma mx};\quad R = \omega^2\sqrt{\{(\Sigma mx)^2 + (\Sigma my)^2\}}.$$

If $\bar{x}$ and $\bar{y}$ be the co-ordinates x and y for the centre of gravity, h the distance of this centre from the axis, and M the whole mass, we have

$$\tan.\rho = \frac{\bar{y}}{\bar{x}},\quad R = \omega^2 M\sqrt{(\bar{x}^2 + \bar{y}^2)} = \omega^2 Mh.$$

Hence, the force R passes through the axis and through the centre of gravity; and its magnitude is the same as if the whole mass were collected in the centre of gravity.

If the axis pass through the centre of gravity, the force R is 0.

The two latter equations give

$$\tan.\sigma = \frac{\Sigma myz}{\Sigma mxz};\quad Sf = \omega^2\sqrt{\{(\Sigma mxz)^2 + (\Sigma myz)^2\}}.$$

It appears by this result, that we cannot determine the force S, or the distance f, but only their product, or the moment of the forces in SF and $S'F'$ to turn the axis round C.

Cor. 1. Hence, we may assign to S any value, and find the corresponding distance f, and *vice versâ*. As we suppose f larger and larger, S becomes smaller and smaller, and when f is indefinitely large, S is indefinitely small. Hence, in this case the force in SF would not affect the value of R, even if it were not neutralized by that in $S'F'$; and we may suppose the two portions $\frac{1}{2}S$ and $\frac{1}{2}S$ to act at the same point F.

Thus it appears, that forces which tend to give, at the same time, a rotatory motion and a motion of translation, to a system, may be resolved into two;—a finite force, producing the motion of translation, and an indefinitely small force, acting at an indefinitely great distance, to produce the rotation. This latter force, being indefinitely small, would not affect the motion of translation, if it were transferred to the centre of rotation. This resolution is frequently used by Euler.

Cor. 2. The directions of the forces R, S, change as the position of the system changes, and in the case we are considering, these directions revolve uniformly round the axis. So that to keep the axis at rest during a whole revolution, it must oppose the tendency to motion in *every* direction with sufficient force.

Cor. 3. If the system, instead of consisting of separate particles, be a continuous body, we must take the differential coefficient of the mass and the integral of this, instead of the partial mass m, and the sum Σ: we shall then have

$$\tan.\sigma = \frac{\int_x yz\frac{dM}{dx}}{\int_x xz\frac{dM}{dx}}; \quad Sf = \omega^2\sqrt{\left\{\left(\int_x xz\frac{dM}{dx}\right)^2 + \left(\int_x yz\frac{dM}{dx}\right)^2\right\}}.$$

Cor. 4. S cannot $= 0$, except $\int_x xz\frac{dM}{dx} = 0$, $\int_x yz\frac{dM}{dx} = 0$.

Prob. I. *A plane revolves about an axis perpendicular to it; to find the pressure on this axis*, fig. 85.

We may suppose C to be placed at the intersection of the given plane and the axis; we shall then have, for all points of the body, $z = 0$, and consequently $S = 0$.

If G be the centre of gravity, and M represent the mass of the plane, the force R acts in CG, and is $= \omega^2 Mh$; CG being h. The same is true for any body which is symmetrical with respect to the plane xCy.

PROB. II. *A uniform straight line revolves about an axis meeting it at any angle; to find the pressure on the axis;* fig. 86.

Let AB be the line, G its centre of gravity; M any point, Dz the axis, GC, MO perpendiculars upon the axis. $CO = z$, $OM = x$, $CG = h$; and if the angle $ODM = a$, $GM = s$, we shall have $x = h + s \,.\, \text{sin.}\, a$, $z = s \,.\, \text{cos.}\, a$, $y = 0$, $\dfrac{dM}{dx} = \dfrac{ds}{dx}$. And let $GA = GB = a$. Then

$$\int_x xz \frac{dM}{dx} = \int (h + s \,.\, \text{sin.}\, a) \,.\, s \,.\, \text{cos.}\, a \,.\, \frac{ds}{dx}$$

$$= \text{cos.}\, a \left(\frac{hs^2}{2} + \frac{s^3 \,\text{sin.}\, a}{3} \right),$$

and the integral taken from $s = -a$ to $s = a$, gives

$$\text{cos.}\, a \,.\, \text{sin.}\, a \,.\, \frac{2a^3}{3} = \frac{Ma^2}{3} \,\text{sin.}\, a \,.\, \text{cos.}\, a.$$

$$\text{Also, } \int_x yz \frac{dM}{dx} = 0\,; \quad \therefore \; Sf = \frac{Ma^2 \omega^2}{3} \,\text{sin.}\, a \,.\, \text{cos.}\, a.$$

And the effort to turn AB round G is the same as if we had a mass $\dfrac{M}{3}$ placed at A, and revolving round an axis GH parallel to CZ, with the angular velocity of the system.

Besides this, we have $R = Mh\omega^2$ acting at G as before.

Prob. III. *A straight line revolves in any position not meeting the axis; to find the pressure;* fig. 87.

Let G be the centre of gravity, GC a perpendicular on the axis. Let the axis of x be parallel to CG; and let the position of the given line be such, that it makes an angle θ with the plane of xy, and that its projection GK on that plane makes an angle η with x. Also let $GM = s$, $CG = h$, $GA = GB = a$. We shall then have

$$s \sin.\theta = MK = z;$$

$$s \cos.\theta \,.\, \sin.\eta = KL; \quad s \cos.\theta \cos.\eta = GL;$$

$$\therefore\ s \cos.\theta \sin.\eta = y, \quad h + s \cos.\theta \cos.\eta = x,$$

$$\int_x xz\frac{dM}{dx} = \int(h + s\cos.\theta\cos.\eta)\, s \sin.\theta \,.\, \frac{ds}{dx}$$

$$= \frac{hs^2}{2}\sin.\theta + \frac{s^3}{3}\cos.\theta\sin.\theta\cos.\eta + \text{const.}$$

which, taken from $s = -a$, to $s = a$, gives

$$\frac{2a^3}{3}\cos.\theta\sin.\theta\cos.\eta,$$

$$\int_x yz\frac{dM}{dx} = \int_x s\cos.\theta\sin.\eta \,.\, s \,.\, \sin.\theta \,.\, \frac{ds}{dx}$$

$$= \frac{2a^3}{3}\cos.\theta \,.\, \sin.\theta \,.\, \sin.\eta,$$

$$\tan.\sigma = \tan.\eta;$$

$$\text{Hence,}\ Sf = \frac{2a^3\omega^2}{3}\,.\,\sin.\theta\,.\,\cos.\theta = \frac{Ma^2\omega^2}{3}\,.\,\sin.\theta\cos.\theta.$$

If from C a perpendicular CT be let fall on KG; the axis will be drawn in the direction CG by the force R, and at the same time will have an effort to turn round the axis CT by the force S.

Sect. II. A BODY REVOLVING ACTED ON BY ANY FORCES.

243. PROP. *Let any forces act upon a system moving round a fixed axis; to find the pressure on the axis.*

Let the forces at each point be resolved in the directions of the co-ordinates; let x_1, y_1, z_1, x_2, y_2, z_2, &c. be the co-ordinates of their points of application; and suppose

X_1, X_2, &c. the parts of the forces parallel to x,
Y_1, Y_2, &c. .... to y,
Z_1, Z_2, &c. .... to z.

The pressures, and the forces which support the axis, may still be reduced to two, acting at different points of the axis, but not in planes perpendicular to it. For the sake of simplicity, let one of these forces act at the origin C. We may resolve each of these forces in two directions; parallel to the axis, and perpendicular to it. Let U, V be the forces perpendicular to the axis, and T the sum of the forces in the axis. Let U pass through the origin, and make an angle ϕ with x; let V act at a distance b from the origin along the axis, and let it make an angle ψ with the line of x; and let the symbol Σ be used to represent the sum of all the products with respect to X_1, X_2, &c. We shall then have for the whole *impressed forces*, (U, V, T, acting to diminish x, y, z,)

$$\Sigma X - U\cos.\phi - V\cos.\psi, \text{ parallel to } x,$$
$$\Sigma Y - U\sin.\phi - V\sin.\psi, \text{ .... to } y,$$
$$\Sigma Z - T \text{ .... to } z.$$

And their moments

$$Vb\sin.\psi + \Sigma(Zy - Yz) \text{ with respect to } Cx,$$
$$Vb\cos.\psi + \Sigma(Zx - Xz) \text{ .... } Cy,$$
$$\Sigma(Yx - Xy) \text{ .... } Cz:$$

these forces acting to turn the system in directions yz, xz, xy.

The effective forces are the same as in the former section, adding the effective forces which accelerate the motion round the axis. These act upon each particle, and are as the distance from the axis; if F be the force at distance 1, in direction xy, Fr will be the force at a distance r; Fmr will be the moving force, and its resolved parts parallel to Cx and Cy will be $-Fmy$ and Fmx. And the moments about Cx, Cy, and Cz arising from this force will be $-Fmxz$, $Fmyz$, Fmr^2.

Hence we have the whole *effective forces*

$$-\omega^2\Sigma mx - F\Sigma my \text{ parallel to } x,$$

$$-\omega^2\Sigma my + F\Sigma mx \ldots\ldots\ldots \text{ to } y,$$

$$-0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ to } z;$$

and their moments in directions xy, xz, yz,

$$\omega^2\Sigma myx - F\Sigma mxz \text{ about } Cx,$$

$$\omega^2\Sigma mxz + F\Sigma myz \ldots\ldots\ldots Cy,$$

$$F\Sigma mr^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots Cz.$$

These effective forces inverted (that is, with their signs changed) must produce an equilibrium with the effective forces: (Art. 202.) hence, we have these six equations,

$$\Sigma X - U\cos.\phi - V\cos.\psi + \omega^2\Sigma mx + F\Sigma my = 0,$$

$$\Sigma Y - U\sin.\phi - V\sin.\psi + \omega^2\Sigma my - F\Sigma mx = 0,$$

$$Vb\sin.\psi + \Sigma(Zy - Yz) - \omega^2\Sigma myz + F\Sigma mxz = 0,$$

$$Vb\cos.\psi + \Sigma(Zx - Xz) - \omega^2\Sigma mxz - F\Sigma myz = 0.$$

$$\Sigma Z - T = 0,$$

$$\Sigma(Yx - Xy) - F\Sigma mr^2 = 0.$$

From the last, F is determined; and then the four first equations serve to determine U, ϕ, ψ, and Vb.

Prob. IV. *Let a body which is symmetrical with respect to a plane passing through the centre of gravity, revolve about a horizontal axis, so that this plane may be vertical; the body being acted on by gravity only, to find the pressure on the axis.*

Let C be taken at the point where the plane of symmetry meets the axis. In this case Σmxz, Σmyz each $= 0$, because every positive value of zm will have a corresponding equal negative value. Also if $\bar{x}$ and $\bar{y}$ be the co-ordinates x and y of the centre of gravity, and M the whole mass; $\Sigma mx = M\bar{x}$, $\Sigma my = M\bar{y}$. Let x be vertical, and y horizontal; then $\Sigma Y = 0$, $\Sigma Z = 0$, and $\Sigma X = Mg$. Also, $\Sigma Yx = 0$, $\Sigma Xy = Mg\bar{y}$. Hence, our equations give

$$-Mg\bar{y} - F\Sigma mr^2 = 0,$$

or putting Mk^2 for Σmr^2, and $h \sin.\theta$ for $\bar{y}$, (h being the distance of the centre of gravity from the axis, and θ the angle which h makes with the vertical,)

$$-gh\sin.\theta = Fk^2, \quad F = -\frac{gh\sin.\theta}{k^2}.$$

And the first four equations become

$$Mg - U\cos.\phi - V\cos.\psi + \omega^2 M\bar{x} + FM\bar{y} = 0,$$

$$-U\sin.\phi - V\sin.\psi + \omega^2 M\bar{y} - FM\bar{x} = 0,$$

$$Vb\sin.\psi = 0,$$

$$Vb\cos.\psi = 0.$$

Hence, $Vb = 0$, and we may make $b = 0$, V remaining indeterminate. This is also seen by considering that the whole pressure will obviously pass through the origin. Hence, putting for F its value, and for $\bar{x}$, $\bar{y}$, $h\cos.\theta$, $h\sin.\theta$, we have

$$Mg - U\cos.\phi = -M\left(\omega^2 h\cos.\theta - \frac{h^2}{k^2}g\sin.^2\theta\right),$$

$$\text{or}\quad U\cos.\phi = M\left(\omega^2 h\cos.\theta - \frac{h^2}{k^2}g\sin.^2\theta + g\right),$$

$$\text{and } U\sin.\phi = M\left(\omega^2 h\sin.\theta + \frac{h^2}{k^2}g\sin.\theta\cos.\theta\right).$$

Let the force U at the axis be resolved into two; R in the direction of the line passing through the centre of gravity, and S perpendicular to this line. Then taking the forces in the direction of x and y, we have

$$R\cos.\theta + S\sin.\theta = M\left(\omega^2 h\cos.\theta - \frac{h^2}{k^2}g\sin.^2\theta + g\right),$$

$$R\sin.\theta - S\cos.\theta = M\left(\omega^2 h\sin.\theta + \frac{h^2}{k^2}g\sin.\theta\cos.\theta\right).$$

Multiply by $\cos.\theta$, $\sin.\theta$, and add, and we have

$$R = M\omega^2 h + Mg\cos.\theta.$$

Multiply by $\sin.\theta$, $\cos.\theta$, and subtract, and we have

$$S = Mg\sin.\theta\left(1 - \frac{h^2}{k^2}\right).$$

In the value of R, the first term, $M\omega^2 h$, arises from the centrifugal force, and is the same as if the mass were collected at the centre of gravity. The second, $Mg\cos.\theta$, is the resolved portion of the weight in the direction of a line passing through the centre of gravity. The forces U and V, which support the axis, are supposed to act opposite to the pressures on it. R and S in this case will act upwards, if θ be less than a right angle.

If the body were a line, revolving in a vertical plane about one of its extremities, its length being a, we should have $h = \frac{a}{2}$, $k^2 = \frac{a^2}{3}$. Hence, in this case, $S = \frac{1}{4}Mg\sin.\theta$. And when the line is horizontal, $S = \frac{1}{4}Mg$, or the pressure downwards is one fourth the weight of the body.

PROB. V. *A uniform straight line revolves about a horizontal axis meeting it in any position; to determine the pressure upon the axis.*

In fig. 86, let CG be considered as the direction of gravity; the notation being the same as in Prob. II. of this Chapter.

The forces will be more distinctly conceived by separating them in the following manner.

We have the forces arising from the rotation, calculated in Prob. II. of this Chapter. They are

(1) A force $Mh\omega^2$ acting in CG.

(2) A force whose moment is $\frac{1}{3}Ma^2\omega^2 . \sin. a . \cos. a$ acting to turn zC about C in the direction zA.

We have also the forces arising from gravity. The force of gravity on each point may be resolved parallel to CG, and perpendicular to the plane zCG; and if θ be the angle which this plane makes with a vertical plane, we shall have, arising from the forces parallel to CG,

(3) A force $Mg\cos.\theta$ acting in CG; and we have to find

(4) The pressure arising from the forces perpendicular to the plane zCG.

These pressures will be perpendicular to the plane zCG, and may be represented by Q and S acting on the axis at distances e and f from C. Hence, the impressed forces are $-Q$, $-S$, and $Mg\sin.\theta$, and the moment is $Qe+Sf$ about CG.

Now if F be the effective force at distance 1, acting to diminish θ, we have $F=\frac{gh\sin.\theta}{k^2}$; and at the point M, the differential coefficient of the effective force with respect to s, the length, is $=\frac{gh\sin.\theta}{k^2}x$, and the differential coefficient of its moment round $CG=\frac{gh\sin.\theta}{k^2}xz$.

Observing that $x=h+s\sin.a$, $z=s\cos.a$, we have

$$\int_s x = hs+\frac{s^2}{2}\sin.a = 2ha = Mh,$$

$$\int_s xs = h \cos. a \frac{s^2}{2} + \sin. a \,.\, \cos. a \frac{s^3}{3}$$

$$= \frac{2a^3}{3} \sin. a \,.\, \cos. a = M \,.\, \frac{a^2}{3} \sin. a \cos. a.$$

Hence, the whole effective force is $Mg \,.\, \frac{h^2 \sin. \theta}{k^2}$,

and its moment about CG is $Mg \,.\, \frac{h a^2 \sin. \theta \sin. a \cos. a}{3k^2}$;

therefore,

$$-Q - S + Mg \sin. \theta = Mg \,.\, \frac{h^2 \sin. \theta}{k^2},$$

$$Qe + Sf = Mg \,.\, \frac{h a^2 \sin. \theta \,.\, \sin. a \,.\, \cos. a}{3k^2}.$$

Whence the forces must be determined.

If we suppose Q to be the force acting at the centre C, and Sf to be the moment of the force of torsion, represented by an infinitely small force at an infinite distance, we have $S=0$, $e=0$; whence

$$Q = Mg \sin. \theta \left(1 - \frac{h^2}{k^2}\right),$$

$$Sf = Mg \sin. \theta \,.\, \frac{h a^2 \sin. a \cos. a}{3k^2}$$

These two forces, along with the three before-mentioned, give the whole pressure on the axis. The last force tends to turn sC about CG as an axis. Hence, on the whole, besides the pressure at C, there will be a tendency to turn Cs about a line inclined to the plane sCG.

We may find k by Chapter III.

For Sf we may substitute two equal forces acting at equal finite distances from C in opposite directions. The forces will be $\frac{S}{2}$, $\frac{S}{2}$, and the distances f, Sf being of the proper value as above.

244. Prop. *When a body revolving about an axis is acted on by a single force in a plane perpendicular to the axis, to find the pressure on the axis.*

This is a particular case of the general proposition, and might be so considered; but as it leads immediately to the properties of the centre of percussion, we shall consider it separately.

Let P be the single force, and let it act parallel to the axis of x, at a point of which the co-ordinates are y', z', the axis of revolution being in the direction of z. Let the body be supposed to be at rest, and let the reaction of the axis be resolved into a force S parallel to x, acting at an ordinate z'', and a force T parallel to y acting at an ordinate z'''. Let, as before, F be the effective force which produces rotation about the axis z, at distance 1. Then Frm is the effective moving force of m at distance r; and $-Fym$, and Fxm are the parts parallel to x and y. Also $-Fxzm$ and $Fyzm$ are the moments of these forces about the axes x and y respectively; and Fr^2m is the moment about z. Hence, we have

the *whole effective forces*

$-F\Sigma ym$, $F\Sigma xm$, 0; and their moments

$-F\Sigma xzm$, $F\Sigma yzm$, $F\Sigma r^2m$.

The impressed forces are P, $-S$ and $-T$; and therefore we have

the *whole impressed forces*

parallel to x, $P-S$; to y, $-T$; to z, 0; and their moments about x, Tz'''; about y, $-Pz'+Sz''$; about z, $-Py'$.

$$\text{Hence, } P-S=-F\Sigma ym, \qquad T=-F\Sigma xm;$$
$$-Pz'+Sz''=F\Sigma yzm, \qquad Tz'''=-F\Sigma xzm;$$
$$-Py'=F\Sigma r^2m.$$

$$\text{Hence, } P-S=\frac{P\Sigma ym}{\Sigma r^2m}\cdot y'; \quad S=P\left(1-\frac{y'\Sigma ym}{\Sigma r^2m}\right);$$
$$Sz''=P\left(z'-\frac{y'\Sigma yzm}{\Sigma r^2m}\right).$$

If the body, instead of being at rest, have any angular velocity, this will produce a centrifugal force in each of the parts, the result of which forces will, by the first section of this Chapter, be a pressure on the axis in the line passing through the centre of gravity. Hence, if we do not consider this force, S will in this case be the same as before.

Cor. 1. If the centre of gravity be in the plane yz, we shall have $\Sigma xm = 0$, and $T = 0$. And if the figure be symmetrical with respect to the plane yz, we shall have $\Sigma xzm = 0$, and $Tz''' = 0$. Σxzm will also $= 0$ in many other cases.

Cor. 2. To find at what point P must be applied, in order that the effect of the force S upon the axis may be 0. We must have $S = 0$, and $Sz'' = 0$*;

$$\therefore\ 0 = P\left(1 - \frac{y'\Sigma ym}{\Sigma r^2 m}\right); \quad 0 = P\left(z' - \frac{y'\Sigma yzm}{\Sigma r^2 m}\right).$$

$$\text{Hence, } y' = \frac{\Sigma r^2 m}{\Sigma ym}; \quad z' = \frac{y'\Sigma yzm}{\Sigma r^2 m} = \frac{\Sigma yzm}{\Sigma ym}.$$

245. Prop. *To find the* Centre of Percussion *of any system.*

If a body be *struck*, in a direction perpendicular to the plane passing through the axis and the centre of gravity, the *centre of percussion* is the point at which the impact must take place, so that it may produce no pressure upon the axis.

P, in the last Article, represents any pressure, and therefore, the conclusions will be true of a *sudden* and *violent* pressure, which is an impact.

The impact is perpendicular on the plane yz, in which the centre of gravity is.

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* It does not follow, that if $S=0$, $Sz''=0$; for Sz'' expresses the force of torsion exerted by S, which may be represented by an indefinitely small force acting at an indefinitely great distance. The same is true of T and Tz'''. See p. 246.

And we have, by Cor. 2, to last Article,

$$y' = \frac{\Sigma r^2 m}{\Sigma y m} = \frac{\Sigma r^2 m}{M h},$$

if h be the distance of the centre of gravity from the axis. Hence, the distance of the centre of percussion from the axis is the same as that of the centre of oscillation

$$z' = \frac{\Sigma y z m}{\Sigma y m} = \frac{\Sigma y z m}{M h}.$$

If the system be symmetrical with respect to a plane which is perpendicular to the axis, and passes through the centre of gravity, and if c be the value of z corresponding to this plane, $\Sigma x z m = M h c$, and $z' = h$. Therefore, the centre of percussion will be in the same plane; and of course, in the line passing through the centre of gravity.

In other cases, we must find $\Sigma x z m$ by integrating.

If $\Sigma x z m$ do not vanish, the whole force on the axis will not vanish. There will be a tendency to twist the system round the axis x.

Ex. 1. A semi-parabola ABC, fig. 88, is moveable about its axis AB: to find the point O, at which it must be struck perpendicularly to its plane, that there may be no stress on the axis.

If z be measured along the axis, and y perpendicular to it, we have

$$\Sigma r^2 m = \int_y \int_z y^2 = \int_z \frac{y^3}{3};$$

and y is to be taken from 0 to $\sqrt{(az)}$, a being the parameter;

$$\therefore \ \Sigma r^2 m = \tfrac{1}{3} \int_z a^{\frac{3}{2}} z^{\frac{3}{2}} = \frac{2 a^{\frac{3}{2}} z^{\frac{5}{2}}}{15}.$$

And if b be the whole length AB, this $= \frac{2a^{\frac{3}{2}}b^{\frac{5}{2}}}{15}$.

$$\Sigma yxm = \int_y \int_x yx = \int_x \frac{y^2}{2} x = \tfrac{1}{2} \int_x ax^2 = \frac{ax^3}{6} = \frac{ab^3}{6},$$

$$\Sigma ym = \int_y \int_x y = \int_x \frac{y^2}{2} = \tfrac{1}{2} \int_x ax = \frac{ax^2}{4} = \frac{ab^2}{4},$$

$$y' = \frac{\Sigma r^2 m}{\Sigma ym} = \frac{2a^{\frac{3}{2}}b^{\frac{5}{2}}}{15} \cdot \frac{4}{ab^2} = \frac{8}{15} a^{\frac{1}{2}} b^{\frac{1}{2}},$$

$$x' = \frac{\Sigma yxm}{\Sigma ym} = \frac{ab^3}{6} \cdot \frac{4}{ab^2} = \tfrac{2}{3} b.$$

Hence, we find the point O by making

$$AL = \tfrac{2}{3} AB, \quad LO = \frac{8}{15} BC.$$

Ex. 2. Let ABC be a right-angled triangle,

$$AL = \tfrac{3}{4} AB, \quad LO = \tfrac{1}{2} BC.$$

CHAP. VI.

THE THREE PRINCIPAL AXES OF ROTATION.

246. THERE are in every body or system, three lines so situated, that if the body revolve freely about any one of them, the pressure upon the axis is 0. The same lines also have this property; that the moment of inertia about one of them is greater, and about another less, than it is about any other axis. These three lines are called the *principal axes:* they are all three at right angles to each other. The properties, &c., of these axes will be the subject of the present Chapter.

PROP. *To find the moment of inertia of a system about an axis, passing through the origin of co-ordinates, and making with the axes of* x, y, z, *any angles* α, β, γ[*].

Let Cx, Cy, Cz, fig. 89, be the axes of co-ordinates, and CI the axis of rotation. Let M be any point, and $MO = r$ a perpendicular on CI. Also, let the co-ordinates of the point M be x, y, z, and $CM = D$.

If the angles which CM makes with Cx, Cy, Cz respectively, be α', β', γ': and δ the angle which CM makes with CI, we shall have

$$\cos.\ \delta = \cos.\ \alpha \,.\, \cos.\ \alpha' + \cos.\ \beta \,.\, \cos.\ \beta' + \cos.\ \gamma \,.\, \cos.\ \gamma'$$ [†].

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[*] The equation $\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1$, connects α, β, γ.

[†] Let a sphere, fig. 90, descrided with centre C, meet Cx, Cy, Cz, CM, CI, in x, y, z, M, I. Then, arcs Ix, Iy, Iz, will be α, β, γ; Mx, My, Mz, will be α', β', γ'; MD will be δ. And we shall have, since xzy is a right angle, and xz a quadrant,

$$\cos.\ Izx = \frac{\cos.\ \alpha}{\sin.\ \gamma};\quad \sin.\ Izx = \cos.\quad Izy = \frac{\cos.\ \beta}{\sin.\ \gamma};$$

$$\cos.\ Mzx = \frac{\cos.\ \alpha'}{\sin.\ \gamma'};\ \sin.\ Mzx = \cos.\ Mzy = \frac{\cos.\ \beta'}{\sin.\ \gamma'};$$

$$\therefore\ \cos.\ IzM = \frac{\cos.\ \alpha \,.\, \cos.\ \alpha' + \cos.\ \beta \,.\, \cos.\ \beta'}{\sin.\ \alpha \,.\, \sin.\ \alpha'}.$$

$$\begin{aligned} \text{And, } \cos.\ \delta &= \cos.\ Izm \,.\, \sin.\ \alpha \,.\, \sin.\ \alpha' + \cos.\ \alpha \,.\, \cos.\ \alpha' \\ &= \cos.\ \alpha \,.\, \cos.\ \alpha' + \cos.\ \beta \,.\, \cos.\ \beta' + \cos.\ \gamma \,.\, \cos.\ \gamma'. \end{aligned}$$

But, $\cos. \alpha' = \dfrac{x}{D}$, &c.;

$$\therefore \cos. \delta = \frac{x}{D}\cos. \alpha + \frac{y}{D}\cos. \beta + \frac{z}{D}\cos. \gamma.$$

Also, $r^2 = D^2 \sin.^2 \delta = D^2 - D^2 \cos.^2 \delta$; and $D^2 = x^2 + y^2 + z^2$;

$$\therefore r^2 = x^2 + y^2 + z^2$$
$$- x^2 \cos.^2 \alpha - y^2 \cos.^2 \beta - z^2 \cos.^2 \gamma$$
$$- 2yz \cos. \beta \cos. \gamma - 2xz \cos. \alpha . \cos. \gamma - 2xy \cos. \alpha \cos. \beta.$$

Putting $x^2 \sin.^2 \alpha$ for $x^2 - x^2 \cos.^2 \alpha$, &c., multiplying by m, the particle at M, and taking the sum indicated by Σ, we have

$$\Sigma r^2 m = \sin.^2 \alpha \Sigma x^2 m + \sin.^2 \beta \Sigma y^2 m + \sin.^2 \gamma \Sigma z^2 m$$
$$- 2 \cos. \beta \cos. \gamma \Sigma yzm - 2 \cos. \alpha \cos. \gamma \Sigma xzm - 2 \cos. \alpha \cos. \beta \Sigma xym.$$

If we put

$$\Sigma x^2 m = f, \quad \Sigma y^2 m = g, \quad \Sigma z^2 m = h,$$
$$\Sigma yzm = F, \quad \Sigma xzm = G, \quad \Sigma xym = H,$$

we have $\Sigma r^2 m = f \sin.^2 \alpha + g \sin.^2 \beta + h \sin^2 . \gamma$

$$- 2F \cos. \beta \cos. \gamma - 2G \cos. \alpha \cos. \gamma - 2H \cos. \alpha \cos. \beta.$$

If the three quantities F, G, H, be each equal to 0, the expression will be simplified. We shall shew in the next Article, that it is always possible to place the axes of x, y, z, in such a position, that F, G, H, shall = 0. In that case,

$$\Sigma r^2 m = f \sin.^2 \alpha + g \sin.^2 \beta + h \sin.^2 \gamma.$$

247. PROP. *To find the position of three axes of rectangular co-ordinates* x′, y′, z′, such that

$$\Sigma y'z'm = 0, \quad \Sigma x'z'm = 0, \quad \Sigma x'y'm = 0.$$

Let x, y, z, be rectangular co-ordinates, having the same origin as x', y', z', and having a known position with respect to the body.

Let x' make with x, y, z, angles, whose cosines are a, b, c,

y' .... a', b', c',

z' .... a'', b'', c''.

Hence, by note, p. 263, the cosine of the angle which x and y make, is $aa'+bb'+cc'$; but this angle is a right angle;

$$\left.\begin{array}{rl} \therefore & aa' + bb' + cc' = 0 \\ \text{Similarly,} & aa'' + bb'' + cc'' = 0 \\ & a'a'' + b'b'' + c'c'' = 0 \\ \text{Also,} & a^2 + b^2 + c^2 = 1 \\ & a'^2 + b'^2 + c'^2 = 1 \\ & a''^2 + b''^2 + c''^2 = 1 \end{array}\right\} \ldots\ldots\ldots\ldots (\beta),$$

by the other note in the same page.

These are six equations of condition.

Now let the moment of inertia be found for any axis which makes angles α, β, γ, with x, y, z: this moment will be

$$f\sin.^2\alpha + g\sin.^2\beta + h\sin.^2\gamma$$
$$-2F\cos.\beta\cos.\gamma - 2G\cos.\alpha\,.\cos.\gamma - 2H\cos.\alpha\,.\cos.\beta.$$

But, if this axis make angles α', β', γ', with x', y', z', and if $\Sigma x'^2 m = X$, $\Sigma y'^2 m = Y$, $\Sigma z'^2 m = Z$; since by supposition, $\Sigma y'z'm = 0$, &c., the moment of inertia will be

$$X\sin.^2\alpha' + Y\sin.^2\beta' + Z\sin.^2\gamma';$$

and these two expressions must be equal.

The latter is equal to

$$X + Y + Z - X\cos.^2\alpha' - Y\cos.^2\beta' - Z\cos.^2\gamma';$$

also $X+Y+Z = \Sigma(x'^2 + y'^2 + z'^2)m = \Sigma D^2 m$, if D be the distance of m from the origin; and this is

$$= \Sigma(x^2 + y^2 + z^2)m = f + g + h.$$

Also, $f \sin.^2 \alpha = f - f \cos.^2 \alpha$, &c.: hence, substituting, the expressions to be made equal, are

$$f \cos.^2 \alpha + \&c. + 2F \cos. \beta \cos. \gamma + \&c.$$

$$\text{and } X \cos.^2 \alpha' + \&c.$$

By the note, p. 263, we have

$$\cos. \alpha' = a \cos. \alpha + b \cos. \beta + c \cos. \gamma,$$

$$\cos. \beta' = a' \cos. \alpha + b' \cos. \beta + c' \cos. \gamma,$$

$$\cos. \gamma' = a'' \cos. \alpha + b'' \cos. \beta + c'' \cos. \gamma.$$

Hence, $X \cos.^2 \alpha' + Y \cos.^2 \beta' + Z \cos.^2 \gamma'$ becomes

$$X(a^2 \cos.^2 \alpha + b^2 \cos.^2 \beta + c^2 \cos.^2 \gamma$$

$$+ 2bc \cos. \beta \cos. \gamma + 2ac \cos. \alpha \cos. \gamma + 2ab \cos. \alpha \cos. \beta)$$

$$+ Y(a'^2 \cos.^2 \alpha + b'^2 \cos.^2 \beta + c'^2 \cos.^2 \gamma$$

$$+ 2b'c' \cos. \beta \cos. \gamma + 2a'c' \cos. \alpha \cos. \gamma + 2a'b' \cos. \alpha \cos. \beta)$$

$$+ Z(a''^2 \cos.^2 \alpha + b''^2 \cos.^2 \beta + c''^2 \cos.^2 \gamma$$

$$+ 2bc'' \cos. \beta \cos. \gamma + 2a''c'' \cos. \alpha \cos. \gamma + 2a''b'' \cos. \alpha \cos. \beta);$$

which must be identical with

$$f \cos.^2 \alpha + g \cos.^2 \beta + h \cos.^2 \gamma$$

$$+ 2F \cos. \beta \,.\, \cos. \gamma + 2G \cos. \alpha \,.\, \cos. \gamma + 2H \cos. \alpha \,.\, \cos. \beta.$$

Equating coefficients of $\cos.^2 \alpha$, &c., we have

$$Xa^2 + Ya'^2 + Za''^2 = f \ldots\ldots (1),$$

$$Xb^2 + Yb'^2 + Zb''^2 = g \ldots\ldots (2),$$

$$Xc^2 + Yc'^2 + Zc''^2 = h \ldots\ldots (3),$$

$$Xbc + Yb'c' + Zb''c'' = F \ldots\ldots (4),$$

$$Xac + Ya'c' + Za''c'' = G \ldots\ldots (5),$$

$$Xab + Ya'b' + Za''b'' = H \ldots\ldots (6),$$

(1) a + (6) b + (5) c, gives

$$Xa(a^2+b^2+c^2)+Ya'(aa'+bb'+cc')+Za''(aa''+bb''+cc'')$$
$$=fa+Hb+Gc;$$

or, by the equations of condition (β),

$$\left.\begin{array}{rl} & Xa=fa+Hb+Gc \\ \text{Similarly, } (2)b+(4)c+(6)a \text{ gives} & Xb=gb+Fc+Ha \\ (3)c+(5)a+(4)b \;\ldots\ldots & Xc=hc+Ga+Fb \end{array}\right\}\ldots(7).$$

From these three equations, and $a^2+b^2+c^2=1$, we must find X, a, b, c.

By the two first, we get

$$(X-f)a-Hb-Gc=0,$$
$$(X-g)b-Fc-Ha=0.$$

Eliminate c;

$$\therefore\ \{(X-f)F+GH\}a-\{(X-g)G+FH\}b=0;$$
$$\therefore\ b=\frac{(X-f)F+GH}{(X-g)G+FH}\cdot a.$$

Eliminate b;

$$\therefore\ \{(X-f)(X-g)-H^2\}a-\{(X-g)G+FH\}c=0;$$
$$\therefore\ c=\frac{(X-f)(X-g)-H^2}{(X-g)G+FH}\,a.$$

Substitute these values in the third of equations (7), viz.

$(X-h)c-Ga-Fb=0$, and we have

$$(X-h)\frac{(X-f)(X-g)-H^2}{(X-g)G+FH}-G-F\frac{(X-f)F+GH}{(X-g)G+FH}=0;$$

or, $(X-f)(X-g)(X-h)$

$$-\{(X-f)F^2+(X-g)G^2+(X-h)H^2\}-2FGH=0;$$

a cubic equation, from which X may be determined.

X being known, we can find a, b, c, by the equations

$$b = \frac{(X-f)F+GH}{(X-g)G+FH}\,a, \quad c = \frac{(X-f)(X-g)-H^2}{(X-g)G+FH}\,a,$$

$$a^2+b^2+c^2=1:$$

and hence the position of the axis x' is determined.

If we make the same combinations as before, of equations (1), (2), (3), (4), (5), (6); only using a', b', c', instead of a, b, c, we shall have the same final result, with the difference of Y instead of X. If we use a'', b'', c'', instead of a, b, c, we shall have Z instead of X in the result. Hence, the same cubic equation will give X, Y, Z; and therefore, these must be its three roots: and hence, there is only one system of three axes, possessing the property required; for the first root of the cubic giving one axis, the second and third roots give the two other axes of the same system.

This cubic has necessarily one root possible; and it may be shewn that the other roots are also possible. If they be impossible, suppose them to be of the form $m+n\sqrt{(-1)}$, and $m-n\sqrt{(-1)}$. The quantities a, b, c, are possible when X is so; and for one of the impossible values of X, the corresponding quantities a', b', c', will be of the form

$$p+q\sqrt{(-1)},\ p'+q'\sqrt{(-1)},\ p''+q''\sqrt{(-1)};$$

and for the other root, a'', b'', c'', they will be of the form

$$p-q\sqrt{(-1)},\ p'-q'\sqrt{(-1)},\ p''-q''\sqrt{(-1)}.$$

Now,

$$a'a''+b'b''+c'c''=0; \quad \therefore\ p^2+p'^2+p''^2+q^2+q'^2+q''^2=0,$$

which cannot be, if p, q, &c., are possible. Therefore, the roots are not of the form supposed.

248. Prop. *Given one principal axis, to find the other two.*

Let the given principal axis coincide with the axis of x. Therefore, $a = 1$, $b = 0$, $c = 0$. And putting these values in the equations (7) in X, a, b, c, we find

$$X = f, \quad 0 = H, \quad 0 = G;$$

the first equation is equivalent to $X - f = 0$: substituting the two latter values in the cubic, dividing out the factor $X - f$, and putting Y for X, we have

$$(Y - g)(Y - h) - F^2 = 0;$$

and the value of Y, obtained from this, must be substituted in the equations

$$Yb' = gb' + Fc + Ha',$$

$$a'^2 + b'^2 + c'^2 = 1.$$

But $H = 0$, and $a' = 0$, since y' must make a right angle with x;

$$\therefore \ (Y - g)b' = Fc'; \ b'^2 + c'^2 = 1.$$

If θ be the angle which y' makes with y, $b' = \cos.\theta$, $c' = \sin.\theta$;

$$\therefore \ \tan.\theta = \frac{c'}{b'} = \frac{Y - g}{F};$$

$$\therefore \ \tan.2\theta = \frac{2F(Y - g)}{F^2 - (Y - g)^2}.$$

$$\text{But } Y^2 - (g + h)Y + gh - F^2 = 0.$$

$$\text{Add } (h - g)Y - (h - g)g = (Y - g)(h - g);$$

$$\therefore \ (Y - g)^2 - F^2 = (Y - g)(h - g);$$

$$\therefore \ \tan.2\theta = \frac{2F}{g - h}:$$

this gives two values of θ, differing by a right angle; and these determine the required positions of y' and z'.

249. Prop. *To find the moment of inertia about any axis, in terms of the moments about the principal axes.*

Let A be the moment about the axis of x: then if r be the distance of a particle m from that axis,

$$r^2 = x^2 + y^2, \quad A = \Sigma r^2 m = \Sigma (x^2 + y^2) m = \Sigma x^2 m + \Sigma y^2 m = g + h.$$

Similarly, if B be the moment about the axis of y, and C about the axis of z, we shall have

$$B = f + h, \quad C = f + g.$$

Now, it is shewn at the end of Art. 246, that if f, g, h, belong to axes for which $\Sigma . yzm = 0$, &c., we shall have, for the moment of inertia (μ), about any other axis,

$$\mu = f \sin.^2 \alpha + g \sin.^2 \beta + h \sin.^2 \gamma.$$

But, since

$$\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1, \quad \sin.^2 \alpha = \cos.^2 \beta + \cos.^2 \gamma;$$

similarly,

$$\sin.^2 \beta = \cos.^2 \alpha + \cos.^2 \gamma, \text{ and } \sin.^2 \gamma = \cos.^2 \alpha + \cos.^2 \beta;$$

and substituting these values,

$$\begin{aligned} \mu &= (g + h) \cos.^2 \alpha + (f + h) \cos.^2 \beta + (f + g) \cos.^2 \gamma \\ &= A \cos.^2 \alpha + B \cos.^2 \beta + C \cos.^2 \gamma. \end{aligned}$$

250. Prop. *Of the moments* A, B, C *about the principal axis, one is greater and another less than the moments about any other axes.*

Let A be the greatest of the three, and C the least, and μ the moment about any axis, and, putting

$$1 - \cos.^2 \beta - \cos.^2 \gamma \text{ for } \cos^2 . \alpha,$$

$$\begin{aligned} \mu &= A (1 - \cos.^2 \beta - \cos.^2 \gamma) + B \cos.^2 \beta + C \cos.^2 \gamma \\ &= A - (A - B) \cos.^2 \beta - (A - C) \cos.^2 \gamma, \end{aligned}$$

and $A - B$, $A - C$ are positive; $\therefore$ μ is less than A.

Similarly, $\mu = A \cos.^2 \alpha + B \cos.^2 \beta + C (1 - \cos.^2 \alpha - \cos.^2 \beta)$

$$= C + (A - C) \cos.^2 \alpha + (B - C) \cos.^2 \beta;$$

$\therefore$ μ is greater than C.

If two of the moments as B, C, be equal

$$\mu = A\cos.^2 \alpha + B(\cos.^2 \beta + \cos.^2 \gamma)$$
$$= A\cos.^2 \alpha + B\sin.^2 \alpha.$$

If all the three A, B, C are equal, $\mu = A$ for every axis.

251. **Prop.** *To find all the axes for which the moments are equal.*

If μ be the moment, all these axes are connected by the equation

$$A\cos.^2 \alpha + B\cos.^2 \beta + C\cos.^2 \gamma = \mu, \text{ a constant quantity,}$$
$$(A - C)\cos.^2 \alpha + (B - C)\cos.^2 \beta = \mu - C.$$

If we suppose a sphere whose radius is 1, to be described with its centre at the origin; and suppose the axis to meet the surface of the sphere in a point, which we may call the *pole* of rotation; the co-ordinates of this pole, parallel to Cx and Cy, are cos. α and cos. β, and these determine the projection of the pole on the plane xy. Hence, the above equation is the equation to the projection, on the plane of xy, of the *locus* of all the poles for which μ is the same, or, as we may call them, the *equi-momental* poles. And it appears by that equation, that this projection is an ellipse with its centre C, and its semi-axes in the direction of x and y, equal to

$$\sqrt{\frac{\mu - C}{A - C}}, \text{ and } \sqrt{\frac{\mu - C}{B - C}};$$

and xy is the plane perpendicular to the axis of least moment.

$$\text{Similarly, } (A - B)\cos.^2 \beta + (A - C)\cos.^2 \gamma = A - \mu,$$

and hence, the projections of the locus of the equi-momental poles, on a plane yz perpendicular to the axis of the greatest moment, are ellipses.

$$\text{Again, } (A - B)\cos.^2 \alpha - (B - C)\cos.^2 \gamma = \mu - B.$$

Hence, the projection on the plane perpendicular to the plane xz of the mean moment is a hyperbola.

Thus, in fig. 91, the curve PQ is the locus of equi-momental poles on the surface of a sphere: and the projection of PQ on the plane xy is the ellipse MN; on the plane yz it is the ellipse RQ, and on the plane xz it is the hyperbola PO.

In fig. 92, are represented the loci of equi-momental poles, on the surface of a sphere concentric with the body. If we make $\mu = B$, we have for the locus, a quadrant of a great circle yB, and for its projection BC, on the plane xz, we have a straight line. When $\mu > B$, we have curves PQ, $P'Q'$, &c. approaching nearer to x as μ is larger. When $\mu < B$, we have curves pq, $p'q'$, &c., approaching to z as μ is smaller.

252. We shall now find the principal axes in given figures.

PROB. *To find the principal axes of a given parallelogram.*

Let, in fig. 93, $AB = 2a$, $AD = 2b$, angle $A = \zeta$. And let the axis of x be perpendicular to the plane at its centre C; y parallel to BA, z perpendicular to BA; $CP = y$, $PM = z$: and let $CH = u$, $HM = v$.

It is manifest, that the axis of x is a principal axis, and we have to find the other two by Art 248. The figure being supposed to be divided into particles by lines parallel to AB and AD; at intervals δu, δv one of these particles at M will be $\delta u \delta v$ sin. ζ. Also, we shall have

$$y = v \cos.\zeta + u, \quad z = v \sin.\zeta,$$

$$g = \int_y y^2 \frac{dM}{dy} = \int_v \int_u y^2 \frac{d^2M}{du\,dv} = \int_v \int_u (v \cos.\zeta + u)^2 \sin.\zeta :$$

and, integrating for u, $= \int_v \frac{1}{3} (u + v \cos.\zeta)^3 \sin.\zeta$,

which taken from $u = -a$ to $u = a$, gives

$$g = \int_v \frac{2a^3 + 6av^2 \cos.^2\zeta}{3} \sin.\zeta$$

$$= \frac{2a^3 v \sin.\zeta}{3} + \frac{2av^3 \sin.\zeta \cos.^2\zeta}{3}.$$

This taken from $v = -b$ to $v = b$, gives

$$g = 4ab \sin.\zeta . \frac{a^2 + b^2 \cos.^2\zeta}{3}.$$

Also $h = \int_z z^2 \frac{dM}{dz} = \int_v \int_u z^2 \frac{d^2M}{du\,dv} = \int_v \int_u v^2 \sin.^3\zeta = 2a \sin.^3\zeta \int_v v^2$

$$= \frac{4ab^3 \sin.^3\zeta}{3} = 4ab \sin.\zeta . \frac{b^2 \sin.^2\zeta}{3}.$$

And $F = \int_y yz \frac{dM}{dy} = \int_v \int_u yz \frac{d^2M}{du\,dv} = \int_v \int_u (u + v \cos.\zeta)\, v \sin.^2\zeta$,

which is, integrating for u, $= \int_v \frac{1}{2} (v \cos.\zeta + u)^2\, v \sin.^2\zeta$;

this taken from $u = -a$ to $u = a$, gives

$$F = \int_v 2av^2 \cos.\zeta \sin.^2\zeta$$

$$= 2a \cos.\zeta \sin.^2\zeta \int v^2 dv = \frac{4ab^3 \cos.\zeta . \sin.^2\zeta}{3}$$

$$= 4ab \sin.\zeta . \frac{b^2 \sin.\zeta \cos.\zeta}{3}.$$

Hence, $\tan.2\theta = \frac{2F}{g-h} = \frac{2b^2 \sin.\zeta \cos.\zeta}{a^2 + b^2 \cos.^2\zeta - b^2 \sin.^2\zeta}$

$$= \frac{b^2 \sin.2\zeta}{a^2 + b^2 \cos.2\zeta},$$

which gives two values of θ, corresponding to the two principal axes in the plane of the parallelogram.

To find the momentum with respect to one of these principal axes, we have by the formula, Art. 246, observing that

$$f = \int_x x^2 \frac{dM}{dx} = 0, \quad \alpha = \frac{\pi}{2}, \quad \beta = \theta, \quad \gamma = \frac{\pi}{2} - \theta,$$

$$\Sigma r^2 m, \text{ or } \int_r r^2 \frac{dM}{dr} = g \sin.^2\theta + h \cos.^2\theta - 2F \sin.\theta \cos.\theta$$

$$= 4ab \sin.\zeta \left\{ \frac{a^2 + b^2 \cos.^2\zeta}{3} \sin.^2\theta + \frac{b^2 \sin.^2\zeta}{3} \cos.^2\theta \right.$$

$$\left. - \frac{2b^2 \sin.\zeta . \cos.\zeta}{3} \sin.\theta \cos.\theta \right\}$$

$$= \frac{4ab \sin.\zeta}{3} \left\{ a^2 \sin.^2\theta + b^2 (\cos.\zeta \sin.\theta - \cos.\theta \sin.\zeta)^2 \right\}$$

$$= \frac{4ab \sin.\zeta}{3} \left\{ a^2 \sin.^2\theta + b^2 \sin.^2 (\zeta - \theta) \right\}$$

$$= \frac{2ab \sin.\zeta}{3} \left\{ a^2 (1 - \cos. 2\theta) + b^2 \left[1 - \cos. 2 (\zeta - \theta)\right] \right\}.$$

Now, $a^2 \cos. 2\theta + b^2 \cos. 2 (\zeta - \theta)$

$$= a^2 \cos. 2\theta + b^2 \cos. 2\zeta \cos. 2\theta + b^2 \sin. 2\theta . \sin. 2\zeta$$

$$= \cos. 2\theta \left\{ a^2 + b^2 \cos. 2\zeta + b^2 \sin. 2\zeta \tan. 2\theta \right\}$$

$$= \cos. 2\theta \left\{ a^2 + b^2 \cos. 2\zeta + \frac{b^4 \sin.^2 2\zeta}{a^2 + b^2 \cos. 2\zeta} \right\}$$

$$= \cos. 2\theta \left\{ \frac{a^4 + 2a^2b^2 \cos. 2\zeta + b^4}{a^2 + b^2 \cos. 2\zeta} \right\}.$$

Also, $\cos.^2 2\theta = \dfrac{1}{1 + \tan.^2 2\theta} = \dfrac{(a^2 + b^2 \cos. 2\zeta)^2}{a^4 + 2a^2b^2 \cos. 2\zeta + b^4}$;

$$\therefore \int_r r^2 \frac{dM}{dr} = \frac{2ab \sin.\zeta}{3} \left\{ a^2 + b^2 \pm \sqrt{(a^4 + 2a^2b^2 \cos. 2\zeta + b^4)} \right\},$$

and the two values give B and C.

Cor. The two moments B and C together are equal to $4ab \sin.\zeta . \dfrac{a^2 + b^2}{3}$, which is the moment A about the axis x.

This proposition is general for plane figures.

In any symmetrical plane figure, the principal axes are, the axes of symmetry, and the axis perpendicular to the plane.

In a sphere, all the axes are principal axes.

In a cube, the same is true.

In a parallelepiped, the lines perpendicular to the surfaces are principal axes, and the moments are as

$$b^2 + c^2, \quad a^2 + c^2, \quad a^2 + b^2.$$

In any figure of revolution, the axis of revolution is a principal axis, and any line drawn in a plane perpendicular to the axis of revolution through the centre of gravity, is a principal axis.

253. **Prop.** *The principal axes are axes about which the body can revolve permanently.*

By Art. 242, Cor. 4, if the body revolve about an axis z, such that $\int_z xz \frac{dM}{dx} = 0$, $\int_z yz \frac{dM}{dx} = 0$; the effort to *turn* the axis round the centre of gravity of the body will be 0; and therefore if the point of the axis, where the perpendicular from the centre of gravity meets it, be fixed, the axis will be fixed. And if, besides this condition, the axis of rotation pass through the centre of gravity, the pressure on this axis will = 0. Hence, if the axis be not fixed, but the body left to itself, it will still revolve about this axis; the axis z for which $\int_z xz \frac{dM}{dx} = 0$, $\int_z yz \frac{dM}{dx} = 0$, is therefore a *permanent axis of free rotation.* Similarly, if $\int_z xy \frac{dM}{dx} = 0$, y and x are also permanent axes of free rotation.

CHAP. VII.

MOTION OF ANY RIGID BODY ABOUT ITS CENTRE OF GRAVITY.

254. OUR object at present is to obtain the equations for the motion of rotation of any body about its centre of gravity, supposing that point to be fixed. It has already been shewn, that when forces act upon any body, the centre of gravity is the centre about which the rotation, when separated from the motion of translation, does take place; and it will be seen hereafter, that the motion of the centre of gravity will be the same as if the same forces acted immediately on that part.

Let three rectangular axes, *fixed in space*, pass through the centre of gravity C, fig. 122, and let x, y, z, be co-ordinates parallel to these axes. Let three rectangular axes, fixed in the body, and moveable with it, likewise pass through the centre, and let x_1, y_1, z_1, be co-ordinates parallel to these. We shall also for the sake of simplicity suppose these latter axes to be the principal axes of the body.

Let, as in Art. 247,

x make with x_1, y_1, z_1, angles whose cosines a, b, c,
y .... a', b', c',
z .... a'', b'', c'',

since the axes make right angles with each other,

We shall have the following equations of condition, by the same reasoning as in p. 265.

$$\left.\begin{aligned}
a^2 + a'^2 + a''^2 &= 1 \\
b^2 + b'^2 + b''^2 &= 1 \\
c^2 + c'^2 + c''^2 &= 1 \\
ab + a'b' + a''b'' &= 0 \\
ac + a'c' + a''c'' &= 0 \\
bc + b'c' + b''c'' &= 0
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

If D be the distance from the origin C of a point of which the co-ordinates are x, y, z, we have $\frac{x}{D}$ = the cosine of the angle which this distance makes with the axis of x, and similarly for y, and the rest. Hence by the note, p. 263,

$$\frac{x}{D} = a\frac{x_1}{D} + b\frac{y_1}{D} + c\frac{z_1}{D},$$

$$\left.\begin{aligned} \text{or,}\quad x &= ax_1 + by_1 + cz_1 \\ \text{similarly,}\quad y &= a'x_1 + b'y_1 + c'z_1 \\ z &= a''x_1 + b''y_1 + c''z_1 \end{aligned}\right\}\ldots\ldots\ldots\ldots(\delta).$$

In like manner,

$$\left.\begin{aligned} x_1 &= ax + a'y + a''z \\ y_1 &= bx + b'y + b''z \\ z_1 &= cx + c'y + c''z \end{aligned}\right\}\ldots\ldots\ldots\ldots(\epsilon).$$

255. *Preliminary formulæ.*

If we eliminate successively z_1 and y_1 from equations (δ), by the usual process, we find

$$(b'c'' - b''c')x + (b''c - bc'')y + (bc' - b'c)z$$
$$= \{c''(ab' - a'b) + c'(a''b - ab'') + c(ab'' - a''b)\}\, x_1$$
$$= Ax_1, \text{ if we make}$$
$$A = c''(ab' - a'b) + c'(a''b - ab'') + c(a'b'' - a''b').$$

But we have by equations (ϵ),

$$Aax + Aa'y + Aa''z = Ax_1;$$

whence, comparing the two values of Ax_1,

$$b'c'' - b''c' = Aa, \quad b''c - bc'' = Aa', \quad bc' - b'c = Aa''.$$

In like manner if we make

$$B = a''(bc' - b'c) + a'(b''c - bc'') + a(b'c'' - b''c'),$$
$$C = b''(ca' - c'a) + b'(c''a - ca'') + b(c'a'' - c''a'),$$

we have

$$c'a'' - c''a' = Bb,\ c''a - ca'' = Bb',\ ca' - c'a = Bb'',$$

$$a'b'' - a''b' = Cc,\ a''b - ab'' = Cc',\ ab' - a'b = Cc''.$$

Substituting these values in the value of A, we have

$$A = C(b''^2 + b'^2 + b^2) = C, \text{ because } b''^2 + b'^2 + b^2 = 1.$$

In like manner $B = A$.

Hence, $Aa = b'c'' - b''c'$, $Aa' = b''c - bc''$,

from which two equations we have

$$\begin{aligned} A(ab' - a'b) &= b'^2c'' - b'b''c' - b'b''c + b^2c'' \\ &= (b^2 + b'^2 + b''^2)c'' - (bc + b'c' + b''c'')b'' \\ &= c'', \end{aligned}$$

because, by (γ), $b^2 + b'^2 + b''^2 = 1$, $bc + b'c' + b''c'' = 0$.

But by another of the above equations, $ab' - a'b = Ac''$, whence $A(ab' - a'b) = A^2c''$; therefore, $A^2 = 1$, and $A = 1$.

Hence, we have the equations

$$\left.\begin{aligned} a &= b'c'' - b''c',\ a' = b''c - bc'',\ a'' = bc' - b'c \\ b &= c'a'' - c''a',\ b' = c''a - ca'',\ b'' = ca' - c'a \\ c &= a'b'' - a''b',\ c' = a''b - ab'',\ c'' = ab' - a'b \end{aligned}\right\} \ldots\ldots\ldots (\zeta).$$

256. Prop. *When a rigid body moves in any manner whatever, to find the position of the axis of instantaneous rotation.*

Whatever be the motions of the different parts of the body, it may at any instant be considered as moving round some axis: this axis is called *the axis of instantaneous rotation.* (Art. 199.)

By differentiating equations (δ) with respect to t, observing that for a given point, x_1, y_1, z_1, are constant, we have

$$\left.\begin{aligned}\frac{dx}{dt} &= x_1\frac{da}{dt} + y_1\frac{db}{dt} + z_1\frac{dc}{dt}\\ \frac{dy}{dt} &= x_1\frac{da'}{dt} + y_1\frac{db'}{dt} + z_1\frac{dc'}{dt}\\ \frac{dz}{dt} &= x_1\frac{da''}{dt} + y_1\frac{db''}{dt} + z_1\frac{dc''}{dt}\end{aligned}\right\}\ldots\ldots\ldots\ldots(\eta).$$

which give the motions of any point, in consequence of the change of the angles whose cosines are a, b, c, a', &c.

If we put $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, $\frac{dz}{dt} = 0$, we find the points of the system for which the velocity $= 0$; that is, if we make

$$x_1\frac{da}{dt} + y_1\frac{db}{dt} + z_1\frac{dc}{dt} = 0\ldots\ldots\ldots\ldots(1),$$

$$x_1\frac{da'}{dt} + y_1\frac{db'}{dt} + z_1\frac{dc'}{dt} = 0\ldots\ldots\ldots\ldots(2),$$

$$x_1\frac{da''}{dt} + y_1\frac{db''}{dt} - z_1\frac{dc''}{dt} = 0\ldots\ldots\ldots\ldots(3).$$

Now $(1)\,c + (2)\,c' + (3)\,c''$ gives

$$x_1\left(c\frac{da}{dt} + c'\frac{da'}{dt} + c''\frac{da''}{dt}\right) + y_1\left(c\frac{db}{dt} + c'\frac{db'}{dt} + c''\frac{db''}{dt}\right)$$

$$+ z_1\left(c\frac{dc}{dt} + c'\frac{dc'}{dt} + c''\frac{dc''}{dt}\right) = 0:$$

also $c^2 + c'^2 + c''^2 = 1$; $\therefore\ c\frac{dc}{dt} + c'\frac{dc'}{dt} + c''\frac{dc''}{dt} = 0$.

And we shall have similar equations by taking

$(1)\,b + (2)\,b' + (3)\,b''$, and $(1)\,a + (2)\,a' + (3)\,a''$.

Suppose

$$c\frac{db}{dt} + c'\frac{db'}{dt} + c''\frac{db''}{dt} = p,$$

and since $cb + c'b' + c''b'' = 0$,

$$b\frac{dc}{dt} + b'\frac{dc'}{dt} + b''\frac{dc''}{dt} = -c\frac{db}{dt} - c'\frac{db'}{dt} - c''\frac{db''}{dt} = -p.$$

Similarly, we will suppose $a\frac{dc}{dt} + a'\frac{dc'}{dt} + a''\frac{dc''}{dt} = q$;

whence, $c\frac{da}{dt} + c'\frac{da'}{dt} + c''\frac{da''}{dt} = -q$;

and we will suppose $b\frac{da}{dt} + b'\frac{da'}{dt} + b''\frac{da''}{dt} = r$,

whence, $a\frac{db}{dt} + a'\frac{db'}{dt} + a''\frac{db''}{dt} = -r$;

and our equations become

$$py_1 - qx_1 = 0, \quad r_1x_1 - pz_1 = 0, \quad qz_1 - ry_1 = 0.$$

Now p, q, r do not vary with the variation of x_1, y_1, z_1; hence, the points for which the velocity is 0, lie in a straight line, passing through the origin of co-ordinates. Therefore this line is, for an instant, immoveable, and the body revolves round it; and the equations just given are the equations to the *axis of instantaneous rotation*.

257. Prop. *To find the angles which the axis of instantaneous rotation makes with* x_1, y_1, z_1.

If IC, fig. 94, be the axis of instantaneous rotation, and if CN, NM, MI, be x_1, y_1, z_1, IN will be perpendicular to Cx_1, and

$$\cos. ICx_1 = \frac{CN}{CI} = \frac{x_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}$$

$$= \frac{px_1}{\sqrt{(p^2x_1^2 + p^2y_1^2 + p^2z_1^2)}}$$

$$= \frac{p}{\sqrt{(p^2 + q^2 + r^2)}},$$

by the equations just found.

Similarly, $\cos. ICy_1 = \dfrac{q}{\sqrt{(p^2+q^2+r^2)}}$.

$$\cos. ICz_1 = \frac{r}{\sqrt{(p^2+q^2+r^2)}}.$$

258. **Prop.** *The angular velocity about the axis* IC *is*

$$\sqrt{(p^2+q^2+r^2)}.$$

Take a point P in the axis Cz_1 at a distance 1 from C; so that for this point, $x_1 = 0$, $y_1 = 0$, $z_1 = 1$.

Therefore, by (δ), $x = c$, $y = c'$, $z = c''$.

And (velocity)^2 of $P = \dfrac{dx^2}{dt^2} + \dfrac{dy^2}{dt^2} + \dfrac{dz^2}{dt^2} = \dfrac{dc^2}{dt^2} + \dfrac{dc'^2}{dt^2} + \dfrac{dc''^2}{dt^2}$.

Now $-p = b\dfrac{dc}{dt} + b'\dfrac{dc'}{dt} + b''\dfrac{dc''}{dt}$, $q = a\dfrac{dc}{dt} + a'\dfrac{dc'}{dt} + a''\dfrac{dc''}{dt}$,

$$0 = c\frac{dc}{dt} + c'\frac{dc'}{dt} + c''\frac{dc''}{dt}.$$

Adding together the squares of these three equations, and taking account of the equations (β), p. 265, we have

$$p^2 + q^2 = \frac{dc^2}{dt^2} + \frac{dc'^2}{dt^2} + \frac{dc''^2}{dt^2};$$

$$\therefore \text{ velocity of } P = \sqrt{(p^2+q^2)}.$$

Also, if we draw PQ perpendicular on CI,

$$PQ = \sin. ICz_1 = \sqrt{(1 - \cos.^2 ICz_1)}$$

$$= \frac{\sqrt{(p^2+q^2)}}{\sqrt{(p^2+q^2+r^2)}};$$

$$\therefore \text{ angular velocity of } P = \frac{\text{velocity of } P}{PQ} = \sqrt{(p^2 + q^2 + r^2)};$$

and the angular velocity is necessarily the same for all the points, in consequence of the rigidity of the system.

259. PROP. *The quantities* p, q, r *of the last three Articles, are the angular velocity to the rigid body about its axis of instantaneous rotation, resolved parallel respectively to the planes* yz, xz, *and* xy.

If we have a plane perpendicular to the axis of instantaneous rotation, the motion of any point is, for the instant, parallel to this plane; and this velocity, resolved in the direction of the plane yz, will be the whole velocity multiplied into the cosine of the angle contained between the two planes. But the angle between the two planes is equal to the angle contained between two lines perpendicular to them, namely, the axis of instantaneous rotation and the axis of x. And the cosine of this angle is $\frac{p}{\sqrt{(p^2 + q^2 + r^2)}}$ by Art. 257: also the whole angular velocity is $\sqrt{(p^2 + q^2 + r^2)}$ by last Article; therefore the product of the two, namely p, is the angular velocity resolved parallel to the plane yz. And in the same manner it appears that the angular velocity parallel to the planes xz and xy respectively is q, and r.

260. PROP. *A rigid system being acted upon by given forces, to find the equations of its motion in terms of* p, q, r.

By the principle, Art. 201, all the forces impressed must be *statically* equivalent to all the effective forces. Let X, Y, Z be the impressed accelerating forces which act upon any point m, in the directions respectively of the co-ordinates x, y, z. The effective forces in these directions are

$$m\frac{d^2x}{dt^2}, \quad m\frac{d^2y}{dt^2}, \quad m\frac{d^2z}{dt^2},$$

t being the independent variable. Hence, by the general conditions of statical equilibrium (equations (2), Analytical Statics, Art. 28.) we have, using Σ, as before, to express the sum of all the quantities relative to different points of the system,

$$\Sigma . m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = \Sigma . m \, (x Y - y X),$$

$$\Sigma . m \left(z \frac{d^2 x}{dt^2} - x \frac{d^2 z}{dt^2} \right) = \Sigma . m \, (z X - x Z),$$

$$\Sigma . m \left(y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} \right) = \Sigma . m \, (y Z - z Y).$$

Assume, for the sake of conciseness, $\Sigma . m \, (x Y - y X) = R$,

$$\Sigma . m \, (z X - x Z) = Q, \quad \Sigma . m \, (y Z - z Y) = P.$$

Integrate the above three equations with respect to t, observing that

$$x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} = \frac{d \left(x \dfrac{dy}{dt} - y \dfrac{dx}{dt} \right)}{dt};$$

and similarly of the rest. And Σ does not affect this integration; therefore

$$\Sigma . m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \textstyle\int_t R,$$

$$\Sigma . m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) = \textstyle\int_t Q,$$

$$\Sigma . m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) = \textstyle\int_t P.$$

These equations in x, y, z, are to be transformed so as to depend on the co-ordinates x_1, y_1, z_1, which are independent of the time t, the effect of the change of t being to change the quantities, a, b, a', &c., and consequently p, q, r. By the equations (δ) and (η) of Articles 254 and 256, we have

$$x\frac{dy}{dt} - y\frac{dx}{dt}$$

$$= (ax_1 + by_1 + cz_1)\left(x_1\frac{da'}{dt} + y_1\frac{db'}{dt} + z_1\frac{dc'}{dt}\right)$$

$$- (a'x_1 + b'y_1 + c'z_1)\left(x_1\frac{da}{dt} + y_1\frac{db}{dt} + x_1\frac{dc}{dt}\right)$$

$$= x_1^2\left(a\frac{da'}{dt} - a'\frac{da}{dt}\right) + y_1^2\left(b\frac{db'}{dt} - b'\frac{db}{dt}\right) + z_1^2\left(c\frac{dc'}{dt} - c'\frac{dc}{dt}\right)$$

$$+ y_1z_1\left(b\frac{dc'}{dt} - b'\frac{dc}{dt} + c\frac{db'}{dt} - c'\frac{db}{dt}\right)$$

$$+ x_1y_1\left(a\frac{dc'}{dt} - a'\frac{dc}{dt} + c\frac{da'}{dt} - c'\frac{da}{dt}\right)$$

$$+ x_1y_1\left(a\frac{db'}{dt} - a'\frac{db}{dt} + b\frac{da'}{dt} - b'\frac{da}{dt}\right).$$

Hence,

$$\Sigma . m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

is equal to the similar sum for the right-hand side of this equation. But the quantities which multiply x_1^2, y_1^2, z_1^2, &c. depend on the time only and are not affected by Σ; so that

$$\Sigma . mx_1^2\left(a\frac{da'}{dt} - a'\frac{da}{dt}\right) = \left(a\frac{da'}{dt} - a'\frac{da}{dt}\right)\Sigma . mx_1^2;$$

and similarly of the other terms. Also, since x_1, y_1, z_1, are the principal axes of rotation, we have

$$\Sigma . my_1z_1 = 0, \quad \Sigma . mx_1z_1 = 0, \quad \Sigma . mx_1y_1 = 0.$$

Hence,

$$\Sigma . m \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

$$= \left(a\frac{da'}{dt} - a'\frac{da}{dt}\right) \Sigma . mx_1^2 + \left(b\frac{db'}{dt} - b'\frac{db}{dt}\right) \Sigma . my_1^2$$

$$+ \left(c\frac{dc'}{dt} - c'\frac{dc}{dt}\right) \Sigma . mz_1^2.$$

Now the quantities $a\frac{da'}{dt} - a'\frac{da}{dt}$ may be reduced by the help of the Preliminary formulæ (ζ) Article 255. Putting for a and a' the values there given, we have

$$a\frac{da'}{dt} - a'\frac{da}{dt} = b'c''\frac{da'}{dt} - b''c'\frac{da'}{dt} - b''c\frac{da}{dt} + bc''\frac{da}{dt}$$

$$= c''\left(b''\frac{da''}{dt} + b'\frac{da'}{dt} + b\frac{da}{dt}\right) - b''\left(c''\frac{da''}{dt} + c'\frac{da'}{dt} + c\frac{da}{dt}\right).$$

But since by Art. 256,

$$r = b''\frac{dc''}{dt} + b'\frac{da'}{dt} + b\frac{da}{dt}, \quad -q = c''\frac{da''}{dt} + c'\frac{da'}{dt} + c\frac{da}{dt},$$

we have

$$a\frac{da'}{dt} - a'\frac{da}{dt} = c''r + b''q.$$

In like manner,

$$b\frac{db'}{dt} - b'\frac{db}{dt} = a''p + c''r,$$

$$c\frac{dc'}{dt} - c'\frac{dc}{dt} = b''q + a''p.$$

Hence,

$$\Sigma . m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = (c''r + b''q)\Sigma . mx_1^2 + (a''p + c''r)\Sigma . my_1^2$$

$$+ (b''q + a''p)\,\Sigma\,.\,mz_1^2$$

$$= a''p\,(\Sigma\,.\,my_1^2 + \Sigma\,.\,mz_1^2) + b''q\,(\Sigma\,.\,mx_1^2 + \Sigma\,.\,mz_1^2)$$

$$+ c''r\,(\Sigma\,.\,mx_1^2 + \Sigma my_1^2)$$

$$= Aa''p + Bb''q + Cc''r;$$

where $A = \Sigma my_1^2 + \Sigma\,.\,mz_1^2 = \Sigma\,.\,m\,(y_1^2 + z_1^2)$ = the moment of inertia about the axis of x; and in like manner B and C are the moments of inertia about the axis of y and z respectively.

Reducing in the same manner the other similar expressions which occur in the above three equations, they become

$$\left.\begin{aligned} Aa''p + Bb''q + Cc''r &= \textstyle\int_t R \\ Aa'p + Bb'q + Cc'r &= \textstyle\int_t Q \\ Aap + Bbq + Ccr &= \textstyle\int_t P \end{aligned}\right\}.$$

In order to get rid of a, b, c, &c. in these equations, differentiate them with respect to t, and add them together, multiplying the first by a'', the second by a', and the third by a.

By what precedes, we have these equations,

$$a''^2 + a'^2 + a^2 = 1, \quad a''\frac{da''}{dt} + a'\frac{da'}{dt} + a\frac{da}{dt} = 0,$$

$$a''b'' + a'b' + ab = 0, \quad a''\frac{db''}{dt} + a'\frac{db'}{dt} + a\frac{db}{dt} = -r,$$

$$a''c'' + a'c' + ac = 0, \quad a''\frac{dc''}{dt} + a'\frac{dc'}{dt} + a\frac{dc}{dt} = q.$$

Hence the result of the steps just mentioned will be

$$A\frac{dp}{dt} + (C - B)\,qr = a''R + a'Q + aP.$$

Also, $B\dfrac{dq}{dt} + (A - C)\,pr = b''R + b'Q + bP,$

$$C\frac{dr}{dt} + (B - A)\,pq = c''R + c'Q + cP,$$

will be obtained in a manner exactly similar.

Let
$$a''R + a'Q + aP = L,$$
$$b''R + b'Q + bP = M,$$
$$c''R + c'Q + cP = N;$$

and we have,

$$\left.\begin{aligned} A\frac{dp}{dt} + (C - B)\,qr &= L \\ B\frac{dq}{dt} + (A - C)\,pr &= M \\ C\frac{dr}{dt} + (B - A)\,pq &= N \end{aligned}\right\}\ldots\ldots\ldots\ldots(\theta),$$

which are the equations of motion required.

It appears, by the assumption of P, Q, R, p. 283, that these quantities are the moments of the impressed moving forces about the axes of x, y, z respectively. And the relation just assumed between P, Q, R, and L, M, N being the same as that between x, y, z and x_1, y_1, z_1, it appears that L, M, N are the moments of the impressed moving forces about the axes x_1, y_1, z_1, respectively.

COR. If the motion take place parallel to a fixed plane, the expressions may be simplified. (The plane will be one of the principal planes.)

Let the plane be that of xy, and of $x_1\, y_1$; then, z and z_1 are always in the same direction. Therefore, $c'' = 1$, $c' = 0$, $c = 0$, $a'' = 0$, $b'' = 0$,

$$a^2 + a'^2 = 1,\ ab + a'b' = 0,$$
$$b^2 + b'^2 = 1,$$
$$p = 0,\ q = 0,\ r = b\frac{da}{dt} + b'\frac{da'}{dt}.$$

If ϕ be the angle which x_1 makes with x,

$$a = \cos.\phi,\ a' = \sin.\phi\ ;\ b = -\sin.\phi,\ b' = \cos.\phi\ ;$$

$$b\frac{da}{dt} + b'\frac{da'}{dt} = \sin.^2\phi\frac{d\phi}{dt} + \cos.^2\phi\frac{d\phi}{dt};\ r = \frac{d\phi}{dt};$$

and the first of the equations (θ) becomes

$$N = C.\frac{d^2\phi}{dt^2}.$$

This equation may also be deduced independently.

261. Prop. *The quantities* p, q, r, *being known, to determine the position of the body.*

Let the plane xCy, fig. 123, cut x_1Cy_1 in the line NC; and let the position of the body be determined by means of the angles ϕ, ψ, and θ; of which

$\psi = xCN$, the angular distance of the *line of nodes* CN from the fixed axis of co-ordinates Cx;

$\phi = NCx_1$, the angular distance, from the node N, of the axis Cx_1 which moves with the body;

$\theta =$ the inclination of the planes xCy, x_1Cy_1, or of the axis Cz_1 to Cz, the angle θ being supposed to have its origin at Cz.

The angles ϕ and ψ being measured always in the same direction, through a whole circumference, and θ through a half circumference, these three angles entirely determine the position of the body.

Let a sphere be described about the point C with radius 1, and let the points x, y, z, x_1, y_1, z_1, N; be upon its surface; we shall then have

$$x_1N = \phi,\ xN = \psi,\ z_1z = \theta;\ \text{also angle } N = \theta;$$

and, Nz_1 Nz_1, quadrants.

Also, joining every two of the points x, y, z, x_1, y_1, z_1, by arcs of great circles, we shall have the following equations by the properties of spherical triangles,

$a = \cos. x_1 x = \cos.\theta \sin.\phi \sin.\psi + \cos.\phi \cos.\psi$, by triangle $x_1 N x$,

$b = \cos. y_1 x = \cos.\theta \cos.\phi \sin.\psi - \sin.\phi \cos.\psi$ .... $y_1 N x$,

$c = \cos. z_1 x = \sin.\theta \sin.\psi$;

for in triangle $z_1 N x$, the angle $z_1 N x$ is the complement of θ;

$a' = \cos. x_1 y = \cos.\theta \sin.\phi \cos.\psi - \cos.\phi \sin.\psi$, by triangle $x_1 N y$,

$b' = \cos. y_1 y = \cos.\theta \cos.\phi \cos.\psi + \sin.\phi \sin.\psi$ .... $y_1 N y$,

$c' = \cos. z_1 y = \sin.\theta \cos.\psi$;

for in triangle $z_1 N y$, the angle $z_1 N y$ is the complement of θ;

$a'' = \cos. x_1 z = -\sin.\theta \sin.\phi$, by triangle $x_1 N z$, where $x_1 N z$ is $\frac{\pi}{2} + \theta$,

$b'' = \cos. y_1 z = -\sin.\theta \cos.\phi$, by triangle $y_1 N z$,

$c'' = \cos. z_1 z = \cos.\theta$.

Hence, $-p = b\dfrac{dc}{dt} + b'\dfrac{dc'}{dt} + b''\dfrac{dc''}{dt}$

$$= \{\cos.\theta \cos.\phi \sin.\psi - \sin.\phi \cos.\psi\} \left\{\cos.\theta \sin.\psi \frac{d\theta}{dt} + \sin.\theta \cos.\psi \frac{d\psi}{dt}\right\}$$

$$+ \{\cos.\theta \cos.\phi \cos.\psi + \sin.\phi \sin.\psi\} \left\{\cos.\theta \cos.\psi \frac{d\theta}{dt} - \sin.\theta \sin.\psi \frac{d\psi}{dt}\right\}$$

$$+ \sin.\theta \cos.\phi \sin.\theta . \frac{d\theta}{dt}$$

$$= \cos.^2\theta . \cos.\phi \frac{d\theta}{dt} - \sin.\theta \sin.\phi \frac{d\psi}{dt} + \sin.^2\theta \cos.\phi \frac{d\theta}{dt}$$

$$= \cos.\phi \frac{d\theta}{dt} - \sin.\theta \sin.\phi \frac{d\psi}{dt};$$

$$\therefore\ p = \sin.\theta \sin.\phi \frac{d\psi}{dt} - \cos.\phi \frac{d\theta}{dt}.$$

Similarly,

$$q = a\frac{dc}{dt} + a'\frac{dc'}{dt} + a''\frac{dc''}{dt} = \sin.\theta \cos.\phi \frac{d\psi}{dt} + \sin.\phi \frac{d\theta}{dt}:$$

Also, we must reduce

$$r = b\frac{da}{dt} + b'\frac{da'}{dt} + b''\frac{da''}{dt}.$$

$$\text{Now, } \frac{da}{dt} = -\sin.\theta \sin.\phi \sin.\psi \frac{d\theta}{dt}$$

$$+ \left\{\cos.\theta \cos.\phi \sin.\psi - \sin.\phi \cos.\psi\right\} \frac{d\phi}{dt}$$

$$+ \left\{\cos.\theta \sin.\phi \cos.\psi - \cos.\phi \sin.\psi\right\} \frac{d\psi}{dt};$$

$$\therefore\ b\frac{da}{dt}$$

$$= -\left\{\cos\ \theta \cos.\phi \sin.\psi - \sin.\phi \cos.\psi\right\}.\sin.\theta.\sin.\phi.\sin.\psi \frac{d\theta}{dt}$$

$$+ \left\{\cos.\theta \cos.\phi \sin.\psi - \sin.\phi \cos.\psi\right\}^2 \frac{d\phi}{dt}$$

$$+ \left\{\cos.\theta \sin.\phi \cos.\psi - \cos.\phi \sin.\psi\right\} \times \left\{\cos.\theta \cos.\phi \sin.\psi - \sin.\phi \cos.\psi\right\} \frac{d\psi}{dt}.$$

And we shall have $b'\frac{da'}{dt}$ by putting $\psi + \frac{\pi}{2}$ for ψ in this expression.

$$\text{Hence, } b\frac{da}{dt} + b'\frac{da'}{dt}$$

$$= -\sin.\theta \cos.\theta \sin.\phi \cos.\phi \frac{d\theta}{dt} + \left\{\cos.^2\theta \cos.^2\phi + \sin.^2\phi\right\} \frac{d\phi}{dt}$$

$$- \left\{\cos.\theta \cos.^2\phi + \cos.\theta \sin.^2\phi\right\} \frac{d\psi}{dt}.$$

Also, $b''\frac{da''}{dt}$

$$= \sin.\theta \cos.\phi . \left\{\cos.\theta . \sin.\phi . \frac{d\theta}{dt} + \sin.\theta \cos.\phi \frac{d\phi}{dt}\right\}.$$

$$\text{Hence, } r = b\frac{da}{dt} + b'\frac{da'}{dt} + b''\frac{da''}{dt} = \frac{d\phi}{dt} - \cos.\theta \frac{d\psi}{dt}.$$

Hence, having found p, q, r, we must determine ϕ, ψ, θ, by means of the equations

$$\left.\begin{aligned} p &= \sin.\phi\sin.\theta\frac{d\psi}{dt} - \cos\phi\frac{d\theta}{dt} \\ q &= \cos.\phi\sin.\theta\frac{d\psi}{dt} + \sin.\phi\frac{d\theta}{dt} \\ r &= \frac{d\phi}{dt} - \cos.\theta\frac{d\psi}{dt} \end{aligned}\right\}\dots\dots\dots(\iota).$$

And ϕ, ψ, θ being known, the position of the body is completely determined.

The moments L, M, N, may be functions of ϕ, ψ, θ: in this case, the six equations (θ) and (ι), will determine the quantities ϕ, ψ, θ, p, q, r.

262. Prop. *A body revolves about its centre of gravity acted upon by no forces; it is required to integrate the equations already found.*

Take the equations (θ), which in this case become

$$\left.\begin{aligned} C\frac{dx}{dt} + (B - A)pq &= 0 \\ B\frac{dq}{dt} + (A - C)pr &= 0 \\ A\frac{dp}{dt} + (C - B)qr &= 0 \end{aligned}\right\}\dots\dots\dots(\kappa).$$

Multiply by r, q, p respectively, and add, and we have

$$Cr\frac{dr}{dt} + Bq\frac{dq}{dt} + Ap\frac{dp}{dt} = 0;$$

$$\therefore\ Cr^2 + Bq^2 + Ap^2 = h^2\dots\dots\dots(\lambda),$$

h being a constant quantity.

Again, multiply equations (κ), by Cr, Bq, and Ap respectively, and add, and we have

$$C^2 r\frac{dr}{dt} + B^2 q\frac{dq}{dt} + A^2 p\frac{dp}{dt} = 0;$$

$$\therefore\ C^2r^2 + B^2q^2 + A^2p^2 = k^2, \text{ a constant quantity}\ldots\ldots(\mu).$$

Again, multiply by c, b, a, and add; whence

$$C\left\{c\frac{dr}{dt} + (qa - pb)r\right\} +$$

$$B\left\{b\frac{dq}{dt} + (pc - ra)q\right\} + A\left\{a\frac{dp}{dt} + (rb - qc)p\right\} = 0.$$

Take the three equations

$$a\frac{dc}{dt} + a'\frac{dc'}{dt} + a''\frac{dc''}{dt} = q,$$

$$b\frac{dc}{dt} + b'\frac{dc'}{dt} + b''\frac{dc''}{dt} = -p,$$

$$c\frac{dc}{dt} + c'\frac{dc'}{dt} + c''\frac{dc''}{dt} = 0;$$

multiplying the first by a, the second by b, the third by c, and add them; and we find, taking account of the equations of condition,

$$\frac{dc}{dt} = aq - bp.$$

In the same manner we shall find

$$\frac{db}{dt} = cp - ar,\quad \frac{da}{dt} = br - cp;$$

and so for the other similar quantities.

Hence the above equation is

$$\therefore\ C\cdot\frac{d.rc}{dt} + B\frac{d.qb}{dt} + A\cdot\frac{d.pa}{dt} = 0,$$

$$\left.\begin{array}{ll}\text{whence} & Crc + Bqb + Apa = l \\ \text{Similarly,} & Crc' + Bqb' + Apa' = l' \\ & Crc'' + Bqb'' + Apa'' = l''\end{array}\right\}\ldots\ldots\ldots(\nu).$$

If we square and add these three, we get

$$C^2r^2 + B^2q^2 + A^2p^2 = l^2 + l'^2 + l''^2 = k^2.$$

263. **Prop.** *When a body revolves, acted on by no forces, there exists a plane, to which it may be referred, which plane is invariable in position.*

If we draw a line mC, making with x, y, z, angles of which the cosines are

$$-\frac{Apa + Bqb + Crc}{\sqrt{(A^2p^2 + B^2q^2 + C^2r^2)}}\ \&\text{c.}:$$

since these quantities are constant by last Article, this line will have the same position during the whole motion of the body, and a plane perpendicular will be fixed.

This plane is called the *Principal Plane of Moments**.

Cor. 1. We may thus find the angles which the line mC makes with x_1, y_1, z_1.

$$\cos. mCx_1 = a.\cos. mCx + a'\cos. mCy + a''.\cos. mCz,$$

$$\cos. mCx_1 = -\frac{Ap}{k};\ \text{similarly,}\ \cos. mCy_1 = -\frac{Bq}{k},$$

$$\cos. mCz_1 = -\frac{Cr}{k}.$$

---

* If we consider at any instant the momentum of each particle of the system; (that is, the product of its velocity and mass;) and if we resolve this momentum parallel to a given plane passing through C, and multiply the resolved part by the perpendicular from C, so as to get its *moment*, we obtain the sum of the moments of the particles referred to this plane. And the plane for which this sum is the greatest, is the plane found above, and hence denominated the Principal Plane of Moments.

Cor. 2. If we take the plane perpendicular to mC for the plane of xy, we have

$$a'' = -\frac{Ap}{k}, \quad b'' = -\frac{Bq}{k}, \quad c'' = -\frac{Cr}{k};$$

or $\sin.\theta \sin.\phi = \dfrac{Ap}{k}$, $\sin.\theta \cos.\phi = \dfrac{Bq}{k}$, $\cos.\theta = -\dfrac{Cr}{k}$.

Prob. I. *All the three moments* A, B, C *being equal, and the body not being acted upon by any forces; to determine its motion.*

In this case the equations (κ) become

$$\frac{dr}{dt} = 0, \quad \frac{dq}{dt} = 0, \quad \frac{dp}{dt} = 0.$$

Hence, p, q, r are constant quantities.

Hence, by Art. 257, the position of the axis of instantaneous rotation is fixed with respect to the body

All the axes of the body possesses the properties of *principal axes:* therefore let the axis of instantaneous rotation coincide with Cx_1; hence, $p = 0$, $q = 0$, $r =$ constant; and equations (ι) becomes

$$\sin.\phi \sin.\theta \frac{d\psi}{dt} - \cos.\phi \frac{d\theta}{dt} = 0,$$

$$\cos.\phi \sin.\theta \frac{d\psi}{dt} + \sin.\phi \frac{d\theta}{dt} = 0,$$

$$\frac{d\phi}{dt} - \cos.\theta \frac{d\psi}{dt} = r,$$

the two first give $\dfrac{d\theta}{dt} = 0$, and $\sin.\theta \dfrac{d\psi}{dt} = 0$, whence $\dfrac{d\psi}{dt} = 0$; $\therefore$ θ and ψ are constant.

Hence, $\dfrac{d\phi}{dt} = r$, a constant quantity.

Therefore the body will revolve about a fixed axis and with a constant velocity.

If the motion have been produced by a single force, acting at a point of the system, (impact or pressure,) the axis of rotation will be perpendicular to the plane which passes through the direction of the force and the fixed centre of gravity.

These conclusions are applicable to a sphere, a cube, &c.

Prob. II. *If two of the moments,* A *and* B, *be equal, and the body be not acted upon by any forces; to determine the motion.*

(This will be the case for all figures of revolution, the axis of revolution coinciding with the axis Cz_1).

The equations (κ) here become

$$C\frac{dr}{dt} = 0,$$

$$A\frac{dq}{dt} + (A - C)pr = 0,$$

$$A\frac{dp}{dt} - (A - C)qr = 0.$$

From the first equation it appears that r is a constant quantity, and the third being differentiated, gives

$$A\frac{d^2p}{dt^2} = (A - C)r\frac{dq}{dt};$$

and substituting from this the value of $\frac{dq}{dt}$ in the second equation, we have, dividing by A^2,

$$\frac{d^2p}{dt^2} + \frac{(A - C)^2r^2}{A^2}.p = 0,$$

whence we easily obtain

$$p = a.\sin.(nt + \gamma);\ \text{putting } n \text{ for } \frac{(A - C)r}{A},$$

and supposing $A > C$; and we have from this

$$q = a \cos. (nt + \gamma);$$

where a and γ are two arbitrary quantities, to be determined from the circumstances of the motion at a given time.

Since r is a constant quantity, and also $p^2 + q^2$ a constant quantity, viz. a^2, it appears from Art. 257, that if IC be the axis of instantaneous rotation, ICx_1 is a constant angle. Hence, the axis of instantaneous rotation describes, with respect to the body, a conical surface about Cx_1.

By substituting the values of p, q, r, just found in equations (ι), p. 291, we obtain

$$a . \sin. (nt + \gamma) = \sin. \phi . \sin. \theta \frac{d\psi}{dt} - \cos. \phi \frac{d\theta}{dt},$$

$$a . \cos. (nt + \gamma) = \cos. \phi . \sin. \theta \frac{d\psi}{dt} + \sin. \phi \frac{d\theta}{dt},$$

$$r = \frac{d\phi}{dt} - \cos. \theta \frac{d\psi}{dt}.$$

Take the plane of xy to be the principal plane of moments; we then have, by Cor. 2 to Art. 263,

$$\sin. \theta \sin. \phi = \frac{Ap}{k} = \frac{Aa}{k} \sin. (nt + \gamma),$$

$$\sin. \theta \cos. \phi = \frac{Bq}{k} = \frac{Aa}{k} \cos. (nt + \gamma),$$

$$\cos. \theta = -\frac{Cr}{k};$$

$$\therefore \tan. \phi = \tan. (nt + \gamma);$$

$$\phi = nt + \gamma; \quad \frac{d\phi}{dt} = n;$$

$$\sin.\theta = \frac{A\alpha}{k}; \ \cos.\theta = -\frac{Cr}{k}; \ \tan.\theta = -\frac{A\alpha}{Cr},$$

and θ is constant. Also by the third of equations (ι)

$$\frac{d\psi}{dt} = \frac{1}{\cos.\theta}\left(\frac{d\phi}{dt} - r\right) = \frac{k(r-n)}{Cr} = \frac{k}{A}.$$

Hence, the body revolves uniformly about Cz_1, while CN, the *line of nodes*, moves uniformly round C in the plane xy.

Suppose, that at the first instant, when $t = 0$, the instantaneous axis CI is in the plane z_1Cx_1* and makes an angle δ with Cz_1; and that the angular velocity about it is ϵ. Then, by the formulæ of Art. 257, since IC is perpendicular to Cy_1 we have

$$\cos.ICz_1 = \cos.\delta = \frac{r}{\sqrt{(p^2+q^2+r^2)}},$$

$$\cos.ICx_1 = \sin.\delta = \frac{p}{\sqrt{(p^2+q^2+r^2)}},$$

$$\cos.ICy_1 = 0 = \frac{q}{\sqrt{(p^2+q^2+r^2)}}; \ \therefore \ q = 0.$$

$$\text{Also,} \ \epsilon = \sqrt{(p^2+q^2+r^2)}.$$

Hence we have, since $t = 0$,

$$\cos.\delta = \frac{r}{\epsilon}, \ \sin.\delta = \frac{\alpha\sin.\gamma}{\epsilon}, \ 0 = \frac{\alpha\cos.\gamma}{\epsilon}; \ \therefore \ \gamma = \frac{\pi}{2},$$

$$\therefore \ r = \epsilon\cos.\delta, \ \alpha = \epsilon\sin.\delta, \ \tan.\theta = -\frac{A}{C}\tan.\delta.$$

$$\text{And } n = \frac{A-C}{A}r = \frac{A-C}{A}.\epsilon\cos.\delta.$$

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* Since any line in the plane x_1Cy_1 is a principal axis, we may take for Cx_1 the intersection of x_1Cy_1 and z_1CI.

Hence, (see Art. 262,) $k^2 = A^2 a^2 + C^2 r^2 = A^2 \epsilon^2 \sin.^2 \delta + C^2 \epsilon^2 \cos.^2 \delta$

$$= \epsilon^2 \left\{A^2 - (A^2 - C^2) \cos.^2 \delta\right\}.$$

Hence, $\dfrac{d\psi}{dt} = \epsilon \sqrt{\left(1 - \dfrac{A^2 - C^2}{A^2} \cos.^2 \delta\right)}$.

If $C = A$, nearly, n is small in comparison of $\epsilon \cos. \delta$; and since $\dfrac{d\phi}{dt} = n$, and $\dfrac{d\psi}{dt} = \epsilon$, nearly, the motion of the body round the axis Cz_1 will be slow, and the motion of Cz_1 about Cz will be comparatively quick.

When ϵ and δ are positive, I is within the quadrantal space $x_1\ y_1\ z_1$, fig. 95; and it appears that in this case θ is negative, and Nx_1y_1, which was in the preceding Articles supposed below Nxy, is here above it.

We have, when $t = 0$, $\phi = \gamma = \dfrac{\pi}{2}$. Hence, N is the pole of z_1x_1; and z, which is a quadrant distant from N, is in z_1x_1: therefore in this case I is in the arc z_1z. And any instant may be considered as that when $t = 0$. Therefore I, the pole of instantaneous rotation is always in z_1z. And since r and $p^2 + q^2$ are constant, CI always makes the same angle with Cz_1.

$$\text{Since } zz_1 = \theta, \text{ and } z_1 I = \delta,$$

we have $zI = \theta + \delta$, $\sin. zI = \sin. \theta \cos. \delta + \cos. \theta \sin. \delta$,

$$\frac{\sin. zI}{\sin. z_1 I} = \cos. \theta \left(\frac{\tan. \theta}{\tan. \delta} + 1\right) = -\frac{Cr}{k}\left(-\frac{A}{C} + 1\right)$$

$$= \frac{(A - C)\, r}{k}.$$

$$\text{Also, } \frac{d\phi}{d\psi} = \frac{nA}{k} = \frac{(A - C)\, r}{k};$$

$$\therefore \sin. z_1 I . \frac{d\phi}{dt} = \sin. zI . \frac{d\psi}{dt}.$$

If with centres z, z_1, we describe circles in fig. 96, on the surface of the sphere, and if $d\phi$, $d\psi$, dt be indefinitely small increments of ϕ, ψ, t; since $d\phi$ is the angle described in dt by the body about z_1 and sin. $z_1 I$ the radius; sin. $z_1 I . d\phi$ will be Im, the arc of the circle which a point would describe in dt. Similarly, $d\psi$ is the angle which the surface $z_1 x_1 y_1$, describes about z; sin. zI is the radius: therefore sin. $zI . d\psi$ is the arc In described by a point about z in dt. Also $Im = In$.

Hence, if the circle Qn be fixed, and if Pm, carried by the pole I of the body, *roll* on Qn; the angular motion will be exactly the same as when the body is left to itself; and this supposition represents the motion of the body in the Problem. The reasonings will be the same if $A < C$. In this case, we shall have $n = \frac{C-A}{A} r$. The point z will be between z_1 and I, and the circle described about z_1 will roll with its interior circumference on the outside of the circle described about z.

PROB. III. *When all the moments are unequal, and the body is acted upon by no forces; to determine the motion.*

Equations (κ) become in this case

$$C \frac{dr}{dt} + (B - A)\, pq = 0,$$

$$B \frac{dq}{dt} + (A - C)\, pr = 0,$$

$$A \frac{dp}{dt} + (C - B)\, qr = 0.$$

Let $pqr = \frac{d\phi}{dt}$, and we have

$$Cr \frac{dr}{dt} + (B - A) \frac{d\phi}{dt} = 0,$$

$$Bq\frac{dq}{dt} + (A - C)\frac{d\phi}{dt} = 0,$$

$$Ap\frac{dp}{dt} + (C - B)\frac{d\phi}{dt} = 0;$$

$$\therefore\ r^2 = \frac{2(A - B)}{C}\phi + c^2,$$

$$q^2 = \frac{2(C - A)}{B}\phi + b^2,$$

$$p^2 = \frac{2(B - C)}{A}\phi + a^2,$$

a, *b*, *c* being the values of p, q, r, when ϕ is 0;

$$\therefore\ \frac{dt}{d\phi} = \frac{1}{pqr}$$

$$= \frac{1}{\sqrt{\left\{\left(2\phi\frac{B - C}{A} + a^2\right)\left(2\phi\frac{C - A}{B} + b^2\right)\left(2\phi\frac{A - B}{C} + c^2\right)\right\}}}.$$

And integrating, we have t in terms of ϕ, and hence ϕ in terms of t. And hence, we have p, q, r; and by Art. 261, the position of the body.

Cor. By Art. 249, we have the moment of inertia at any time

$$\Sigma rm = A\cos.^2\alpha + B\cos.^2\beta + C\cos.^2\gamma$$

$$= \frac{Ap^2 + Bq^2 + Cr^2}{p^2 + q^2 + r^2} = \frac{h^2}{p^2 + q^2 + r^2}, \text{ by Art. 262;}$$

$$\therefore\ (p^2 + q^2 + r^2)\,\Sigma r'^2 m = h^2.$$

Hence, the sum of each particle multiplied into the square of its velocity, continues constant during the motion.

Prob. IV. *A solid body revolves about its centre of gravity, so that its axis of rotation coincides nearly with one of its principal axes; to find the motion.*

If IC be the instantaneous axis, always nearly coinciding with Cx_1,

$$\sin. ICx_1 = \frac{\sqrt{(p^2+q^2)}}{\sqrt{(p^2+q^2+r^2)}};$$

therefore p and q must both be small, $C\frac{dr}{dt} = 0$, nearly, and r is nearly constant, and = the velocity of rotation = n;

$$\therefore\ B\frac{dq}{dt} + (A-C)np = 0,\quad A\frac{dp}{dt} + (C-B)nq = 0;$$

$$\therefore \text{ as before, } p = a\sin.(n't+\gamma),\quad q = \beta\cos.(n't+\gamma),$$

$$\text{where } n' = n\sqrt{\frac{(A-C).(B-C)}{AB}},\quad \beta = a\sqrt{\frac{(A-C).A}{(B-C).B}},$$

and knowing p and q, we might find ϕ, ψ, θ.

Now if IC and x_1C coincide, $a = 0$, and $\beta = 0$. Hence, a and β are small when ICx_1 is small, and if n' be real, p and q will always be small; but if n' be imaginary, p and q become exponentials, and increase beyond small values, and the solution is not applicable. In the first case, the axis will oscillate about Cx_1: in the second case, the axis IC will leave Cx_1, and oscillate about another of the principal axes.

The first case will happen, if $(A-C).(B-C)$ be positive; ∴ if C be the *greatest* or *least* of the moments A, B, C.

The second case will happen if C be the *mean* moment.

Hence, if in the second case the body at first oscillate *accurately* about Cx_1, and if that axis be disturbed ever so little, the axis will entirely leave its position.

Cor. 1. We may prove that p and q cannot increase beyond a certain limit.

$$Ap^2 + Bq^2 + Cr^2 = h^2 \dots (\lambda), \quad A^2p^2 + h^2q^2 + Cr^2 = k^2 \dots (\mu),$$

$$(\mu) - (\lambda)\, C \text{ gives } A(A - C)p^2 + B(B - C)q^2 = k^2 - Ch^2.$$

Hence, if p and q are small at first, and if $A - C$ and $B - C$ are of the same sign, p and q will always remain small.

Cor. 2. The limits of p^2 and q^2 as given by the condition just obtained, are

$$p^2 < \frac{k^2 - Ch^2}{A(A - C)}, \quad q^2 < \frac{k^2 - Ch^2}{B(B - C)}.$$

CHAP. VIII.

MOTION OF A RIGID BODY ACTED ON BY ANY FORCES.

264. WE shall consider, in the present Chapter, the motion of a body, whether it be in free space, or constrained to move upon a given plane. The same principles are applicable in both cases, if we include, among the forces which act upon the body, the reaction of the plane which it touches, and then eliminate this reaction.

In a large class of problems of this kind, one of the principal axes moves parallel to itself, and consequently, all the particles move in planes perpendicular to this axis; for instance, if a body bounded by a cylindrical surface of any form roll upon a plane. We shall take this more simple case separately.

Sect. I. WHEN THE MOTIONS OF ALL THE PARTICLES ARE IN PARALLEL PLANES.

PROP. *The body being acted upon by any forces, the motion of the centre of gravity will be the same as if all those forces acted at the centre.*

By the principle in Article 203, the impressed and effective forces must be equivalent, and their moments about any point must also be equivalent; the former consideration will determine the motion of translation of the centre of gravity, and the latter will determine the motion of rotation about this centre.

If x', y' be the co-ordinates of any particle, the velocity in the direction of x' is $\frac{dx'}{dt}$, and therefore the effective accelerating force is $\frac{d^2x'}{dt^2}$, and the effective moving force $m\frac{d^2x'}{dt^2}$.

Thus we have the whole effective force on m, parallel to $x' = \Sigma . m \dfrac{d^2x'}{dt^2}$; and the whole effective force on m, parallel to $y' = \Sigma . m \dfrac{d^2y'}{dt^2}$. Also, if X and Y be the impressed accelerating forces on any particle m,

the whole impressed forces are ΣmX, $\Sigma m Y$.

Hence we have

$$\Sigma . m \frac{d^2x'}{dt^2} = \Sigma mX, \quad \Sigma . m \frac{d^2y'}{dt^2} = \Sigma m . Y.$$

Let $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre of gravity; and x, y the co-ordinates of m from 'the centre of gravity, so that

$$x' = \bar{x} + x, \quad y' = \bar{y} + y;$$

$$\therefore \ \Sigma mx = 0, \ \Sigma my = 0; \ \therefore \ \Sigma . m \frac{d^2x}{dt^2} = 0, \ \Sigma . m \frac{d^2y}{dt^2} = 0;$$

$$\therefore \ \Sigma . m \frac{d^2x'}{dt^2} = \Sigma . m \frac{d^2\bar{x}}{dt^2} = M . \frac{d^2\bar{x}}{dt^2};$$

(M being Σm, the whole mass;) because $\bar{x}$ is the same for all particles. Similarly,

$$\Sigma . m \frac{d^2y'}{dt^2} = M \frac{d^2\bar{y}}{dt^2};$$

$$\therefore \ \frac{d^2\bar{x}}{dt^2} = \Sigma . \frac{mX}{M}, \ \frac{d^2\bar{y}}{dt^2} = \Sigma . \frac{mY}{M} \ \ldots\ldots\ldots \ (\xi),$$

Which are the equations that would result if all the forces were applied to the centre of gravity.

Cor. The forces mX, mY, are here understood to be pressures; and as impact is only a short pressure, the results are true of impact. The forces mX, mY, are measured by the momenta generated in a time 1″; the force of impact may be measured by the *whole* momentum which it would generate.

265. **Prop.** *The body being acted upon by any forces, the motion of rotation will be affected in the same manner as if the centre of gravity were fixed, and the same forces were applied.*

The moment of the effective force with respect to C is

$$\Sigma . m \left(x' \frac{d^2 y'}{dt^2} - y' \frac{d^2 x'}{dt^2} \right).$$

And putting for x', $\bar{x} + x$, and for y', $\bar{y} + y$, it is

$$\Sigma . m \left\{ \begin{array}{l} \bar{x} \dfrac{d^2 \bar{y}}{dt^2} + \bar{x} \dfrac{d^2 y}{dt^2} + x \dfrac{d^2 \bar{y}}{dt^2} + x \dfrac{d^2 y}{dt^2} \\ - \bar{y} \dfrac{d^2 \bar{x}}{dt^2} - \bar{y} \dfrac{d^2 x}{dt^2} - y \dfrac{d^2 \bar{x}}{dt^2} - y \dfrac{d^2 x}{dt^2} \end{array} \right\}.$$

And observing that

$$\Sigma . m = M, \ \Sigma . mx = 0, \ \Sigma . my = 0, \ \Sigma . m \frac{d^2 x}{dt^2} = 0, \ \Sigma . m \frac{d^2 y}{dt^2} = 0;$$

and that $\bar{x}$ and $\bar{y}$ are not affected by Σ; it becomes

$$M \left(\bar{x} \frac{d^2 \bar{y}}{dt^2} - \bar{y} \frac{d^2 \bar{x}}{dt^2} \right) + \Sigma . m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right).$$

Also the moment of the effective forces is

$$\begin{aligned} \Sigma . m (Yx' - Xy') &= \Sigma . m (Y\bar{x} - X\bar{y} + Yx - Xy) \\ &= M (Y\bar{x} - X\bar{y}) + \Sigma . m (Yx - Xy). \end{aligned}$$

Equating, we have, observing that the terms multiplied by M are equal by last Article,

$$\Sigma m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = \Sigma . m (Yx - Xy); \ldots\ldots\ldots (\pi);$$

which is the equation that would result if the centre of gravity were fixed.

Cor. If in fig. 97, GA be a line always parallel to a line fixed in space, GM a line in the plane of xy, fixed with respect to the body, M any particle, of which the co-ordinates from G are x, y: it will be seen as in Art. 22, that

$$\tfrac{1}{2}\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)$$

is the second differential coefficient of the sector AGM; and hence, since GM is constant, if $GM = r$, $AGM = \theta$, we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = r^2\frac{d^2\theta}{dt^2}.$$

Therefore,

$$\Sigma . mr^2 . \frac{d^2\theta}{dt^2} = \Sigma . m(Yx - Xy).$$

Or if as before, $\Sigma . mr^2 = Mk^2$,

$$\frac{d^2\theta}{dt^2} = \frac{\Sigma . m(Yx - Xy)}{Mk^2} \text{............}(\rho)^*.$$

266. Prop. *To find the centre of spontaneous rotation.*

The centre of spontaneous rotation is the point which remains at rest for an instant when the body is put in motion by any force.

Thus, if a body GC, fig. 98, be acted upon by a force which acts in the line PC, it will, for the first instant, revolve round some point O, which may be thus determined.

Let the force in PC be perpendicular to GC, and $= P$; and the mass of the body being M, the space Gg described by G in a small time t, will be $\frac{P}{M}.\frac{t^2}{2}$, the force being sup-

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[*] These formulæ might be used in solving many of the problems in Chap. II. So long as Mk^2 is the same, the motion will be the same whatever be the form of the body. Thus if, in Prob. V, fig. 12, instead of a straight line PQ, we had a body of any form, of which one point slides along AX, and the other along AY, the motion would be the same as is investigated, p. 63.

posed constant for the time t. (See Art. 264.) And by this motion of translation, any point, as O will be transferred to o, Oo being equal to Gg. Let $GO=l$, $CG=h$. By Art. 265, the effect of the force P in producing rotation will be the same as if the point G were fixed; but in this case by Art. 205, the accelerating force causing O to revolve round G is $\frac{Phl}{Mk^2}$, and the space generated by it in the time t is $\frac{Phl}{Mk^2}\cdot\frac{t^2}{2}$. And if this be equal to Oo, the point O will be carried backwards by the rotation, just as much as it is carried forwards by the translation, and will be absolutely at rest: that is, if

$$\frac{P}{M}\cdot\frac{t^2}{2}=\frac{Phl}{Mk^2}\cdot\frac{t^2}{2},$$

$$\text{or } l=\frac{k^2}{h}.$$

Hence, by (a), Art. 208, C and O are so situated that if one be the centre of suspension, the other is the centre of oscillation, or percussion.

Cor. If the force which acts at C be a force of impact, the point O will be still the same, Gg and Oo being in this case described by the uniform velocities which the impact generates.

Prob. I. *To find at what distance from the Earth's centre a force must have acted, to generate at the same time its progressive and rotatory motion.*

This problem stated generally, is, having given the velocity of the centre of gravity and the angular velocity in a body revolving about a permanent axis; to find at what distance from the centre of gravity a single force would produce them.

Let P be the force acting at a distance h from the center; then $\frac{P}{M}$ is the force which accelerates the centre of gravity, and $\frac{Ph}{Mk^2}$ is the force which accelerates a point at distance 1 about the centre of gravity; and hence, the whole velocities generated in any times by these two forces will be as the forces. If v be the velocity of the centre, and a the angular velocity, measured by the space described by a particle at distance 1, we have

$$\frac{P}{M} : \frac{Ph}{Mk^2} :: v : a; \ \therefore h = \frac{k^2 a}{v}.$$

In the case of the Earth, let its radius be taken $= 1$, and let the unit of time be 1 day: therefore, $a = 2\pi$. And the radius of the Earth's orbit being r, the space described in 366,24 sidereal days is $2\pi r$, and therefore $v = \frac{2\pi r}{366{,}24}$. And since the Earth is a sphere, by p. 232, Ex. 20,

$k^2 = \frac{2}{5}$. Hence, $h = \frac{2}{5} \cdot \frac{366{,}24}{r}$, and $r = 24266$, nearly;

$$\therefore h = \frac{1}{165{,}6}, \text{ nearly.}$$

Hence, we find the centre of spontaneous rotation by the formula to be $l = \frac{k^2}{h} = 66{,}25$, which is a little greater than the Moon's distance from the Earth.

In the same way for Mars we should have $h = \frac{1}{464}$ of his radius, nearly: and for Jupiter, $h = \frac{1}{3}$ of his radius, nearly. In the latter planet, the centre of spontaneous rotation is only $\frac{6}{5}$ of the radius from the centre*.

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* John Bernoulli's Works, Vol. IV. p. 284.

267. The following problems refer to such motions as the *rocking* of a cradle, the oscillation, or as it has been called, *titubation* of a body on a curved base, and the *rolling* of a body not spherical. We shall solve some of them, first neglecting, and afterwards considering, friction.

PROB. II. *An ellipse, with its plane vertical, rocks and slides upon an horizontal plane which is perfectly smooth; to determine the motion*, fig. 99.

Let P be the point of contact with the horizontal plane, $CM = x$, $MP = y$, $CA = a$, $CB = b$, the semi-axes; PN vertical, CN horizontal; and let the angle ACH be θ,

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}\,; \quad \frac{dy}{dx} = -\frac{b}{a}\,\frac{x}{\sqrt{(a^2 - x^2)}}\,;$$

$$\text{but at } P,\ -\frac{dx}{dy} = \tan.\theta\,; \ \therefore\ \frac{\sin.\theta}{\cos.\theta} = \frac{a\sqrt{(a^2 - x^2)}}{bx}\,;$$

$$\text{hence, } x = \frac{a^2\cos.\theta}{\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}}\,; \text{ and } y = \frac{b^2\sin.\theta}{\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}}\,.$$

Hence, if $\bar{y}$ be CH or NP,

$$\bar{y} = x\cos.\theta + y\sin\theta = \sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}.$$

$$\text{Also } CN = x\sin.\theta - y\cos.\theta = \frac{(a^2 - b^2)\sin.\theta\cos.\theta}{\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}}\,.$$

Now let R be the reaction upwards at the point P, and $R.CN$ its moment with respect to C; M the mass of the body, Mg its weight, Mk^2 its moment of inertia; the centre of gravity being supposed to be at C. Then by formulæ (ξ) and (ρ) of Articles 264 and 265,

$$\frac{d^2\bar{y}}{dt^2} = \frac{R}{M} - g,$$

$$\frac{d^2\theta}{dt^2} = \frac{R(a^2 - b^2)\sin.\theta\cos.\theta}{Mk^2\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}}\,.$$

But since $\bar{y} = \sqrt{(a^2 \cos.^2 \theta + b^2 \sin.^2 \theta)}$;

$$\frac{d\bar{y}}{d\theta} = -\frac{(a^2 - b^2)\sin.\theta\cos.\theta}{\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}}:$$

hence the second equation becomes, multiplying by $\frac{d\theta}{dt}$,

$\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = -\frac{R}{Mk^2}.\frac{d\bar{y}}{dt}$. And eliminating R by the first,

$$\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{1}{k^2}\frac{d\bar{y}}{dt}\frac{d^2\bar{y}}{dt^2} + g\frac{1}{k^2}\frac{d\bar{y}}{dt} = 0.$$

Multiply by $2k^2$ and integrate; we have thus

$$k^2\frac{d\theta^2}{dt^2} + \frac{d\bar{y}^2}{dt^2} + 2g\bar{y} = C.$$

Put for $\bar{y}$ and for $\frac{d\bar{y}}{dt}$ their values, and a^2e^2 for $a^2 - b^2$; and this becomes

$$\left\{k^2 + \frac{a^4e^4\sin.^2\theta\cos.^2\theta}{a^2\cos.^2\theta + b^2\sin.^2\theta}\right\}\frac{d\theta^2}{dt^2}$$
$$+ 2g\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)} = C.$$

Hence the angular velocity $\frac{d\theta}{dt}$ is known. C is to be determined by knowing the value of this velocity for a given value of θ.

If when CA is vertical, the angular velocity be α, we have $C = k^2\alpha^2 + 2ga$. And when CA becomes horizontal,

$$k^2\frac{d\theta^2}{dt^2} + 2gb = k^2\alpha^2 + 2ga;$$

$$\therefore \frac{d\theta^2}{dt^2} = \alpha^2 + \frac{2g(a-b)}{k^2};$$

therefore, a point at the distance k from C acquires as much velocity as if it had fallen vertically through the difference of the axes CA, CB.

Since there is no lateral force, the centre of gravity C will ascend and descend in a vertical line. The surface will slide along the horizontal line PQ. If there be so much friction as to prevent this sliding, it will roll; the centre of gravity will have a lateral motion, and the problem will no longer be the same.

The figure may be an elliptical cylinder with a horizontal axis: in that case, $k^2 = \frac{1}{4}(a^2 + b^2)$. Or it may be an elliptical spheroid, APB being one of its principal sections: in this case, $k^2 = \frac{1}{5}(a^2 + b^2)$. It may be any figure in which the vertical ellipse is the part which touches the horizontal line, and the centre of gravity is at C.

If the plane PQ were inclined to the horizon at an angle ϵ, the vertical plane of the ellipse being still perpendicular to the plane PQ, the rotatory motion would be the same, putting $g \cos. \epsilon$ for g. And the body would move along the plane, so that the motion parallel to the plane should be that of a point sliding down an inclined plane. For the part of gravity parallel to the plane could not affect the rotatory motion, since its result would pass through the centre of gravity.

Prob. III. *When the major axis of the ellipse, in last Problem, is nearly horizontal; to determine the small oscillations.*

Let ϕ be the angle which CB makes with the vertical; $\therefore$ $\phi = \frac{\pi}{2} - \theta$. And when ϕ is very small, $\sin.^2\theta = \cos.^2\phi = 1 - \phi^2$, $\cos.^2\theta = \sin.^2\phi = \phi^2$. And, by last Problem, neglecting ϕ^4,

$$\left(k^2 + \frac{a^4 e^4 \phi^2}{b^2}\right) \frac{d\phi^2}{dt^2} = C - 2g\sqrt{(b^2 + a^2 e^2 \phi^2)}$$

$$= C - 2g\left(b + \frac{a^2 e^2 \phi^2}{2b}\right);$$

$$(\text{and if } \phi = \beta \text{ when } \frac{d\phi}{dt} = 0) = \frac{a^2 e^2 g}{b}(\beta^2 - \phi^2).$$

Hence, neglecting $\beta^2\phi^2$,

$$\frac{d\phi^2}{dt^2} = \frac{a^2e^2g}{k^2b}(\beta^2-\phi^2);$$

$$\therefore \text{ if } \frac{k^2b}{a^2e^2} = l,$$

$$\sqrt{\frac{g}{l}} \cdot \frac{1}{\sqrt{(\beta^2-\phi^2)}} = -\frac{dt}{d\phi}; \quad t = -\sqrt{\frac{g}{l}} \cdot \text{arc}\left(\sin. = \frac{\beta}{\phi}\right);$$

and for a whole oscillation, this must be taken from $\phi = \beta$ to $\phi = -\beta$. Hence, if time of an oscillation = T,

$$T = \pi\sqrt{\frac{g}{l}}.$$

It appears from this, that the length of the isochronous pendulum is $l = \frac{k^2b}{a^2e^2}$.

Prob. IV. *Let the body rock and slide on a horizontal plane, the part which comes in contact with the plane being a portion of a common cylinder with its axis horizontal; to determine the motion*, fig. 100.

Let a line be drawn through the centre of the cylinder, and through the centre of gravity; let the latter point be at a distance c from the former; let θ be the angle which this line makes with the vertical; then it will be found by proceeding as in Prob. II, that

$$\frac{d\theta^2}{dt^2} = \frac{2cg(\cos.\theta - \cos.\beta)}{k^2 + c^2\sin.^2\theta};$$

where Mk^2 is the moment of inertia about the centre of gravity.

If the body perform small oscillations, we shall have for the length of the isochronous pendulum, $l = \frac{k^2}{c}$.

In these cases, the centre of gravity ascends and descends vertically, and goes up and down while the body goes once back and forwards.

Prob. V. *A cylindrical body rocks and rolls on a horizontal plane, the friction being such as to prevent all sliding*; *to determine its motion*, fig. 100.

Let C be the centre of the circle AP, G the centre of gravity; P the point of contact with the horizontal plane; PC will be vertical; let GN be horizontal. Let a be the point in which A comes in contact with the plane: then, $AP = aP$.

Let $CA = a$, $CG = c$, $aH = x$, $HG = y$; $ACP = \theta$,

$$x = AP - PH = a\theta - c \sin. \theta,$$

$$y = CP - CN = a - c \cos. \theta.$$

Now, the forces which act upon the body are the force of gravity, and the force which arises from the pressure and the friction at P. Let these forces at P be composed of a horizontal force Q, and a vertical force R. And, by Art. 264, the effect on the motion of the centre of gravity will be the same as if all the forces were applied immediately at that point. Hence,

$$\frac{d^2x}{dt^2} = \frac{Q}{M}, \quad \frac{d^2y}{dt^2} = \frac{R}{M} - g.$$

And, by putting for x and y their values,

$$\frac{d\left(a\dfrac{d\theta}{dt} - c\cos.\theta\dfrac{d\theta}{dt}\right)}{dt} = \frac{Q}{M},$$

$$\frac{d\,.\,c\sin.\theta\dfrac{d\theta}{dt}}{dt} + g = \frac{R}{M}.$$

Also, by Art. 265, the effect of the forces in producing rotation about the centre of gravity, is the same as if it were fixed, and the forces Q, R, acted on the body; hence,

$$\frac{d^2\theta}{dt^2} = -\frac{Q\,.\,GH + R\,.\,GN}{Mk^2} = -\frac{Q(a - c\cos.\theta) + Rc\sin.\theta}{Mk^2}.$$

Substituting for $\frac{Q}{M}$ and $\frac{R}{M}$, and multiplying by $2k^2\frac{d\theta}{dt}$, we find

$$2k^2 . \frac{d\theta}{dt}\,\frac{d^2\theta}{dt^2} + 2\left(a\frac{d\theta}{dt} - c\cos.\theta\frac{d\theta}{dt}\right)\frac{d\left(a\frac{d\theta}{dt} - c\cos.\theta\frac{d\theta}{dt}\right)}{dt}$$

$$+ 2c.\sin.\theta\frac{d\theta}{dt}\,\frac{d.c\sin.\theta\frac{d\theta}{dt}}{dt} + 2gc\sin.\theta\frac{d\theta}{dt} = 0.$$

Integrating,

$$k^2\frac{d\theta^2}{dt^2} + (a - c\cos.\theta)^2.\frac{d\theta^2}{dt^2} + c^2\sin.^2\theta\frac{d\theta^2}{dt^2} - 2gc\cos.\theta = C;$$

$$\text{or, } (k^2 + a^2 + c^2 - 2ac\cos.\theta)\frac{d\theta^2}{dt^2} = 2gc(\cos.\theta - \cos.\beta),$$

β being the value of θ when the velocity is 0.

If we make $k^2 + (a-c)^2 = b^2$, we have

$$\left(b^2 + 4ac.\sin.^2\frac{\theta}{2}\right)\frac{d\theta^2}{dt^2} = 4gc\left(\sin.^2\frac{\beta}{2} - \sin.^2\frac{\theta}{2}\right).$$

If we suppose β and θ small, and neglect θ^4, $\beta^2\theta^2$, &c. we have

$$\frac{d\theta^2}{dt^2} = \frac{gc}{b^2}(\beta^2 - \theta^2),$$

whence it appears, as in Prob. III, that the length of the isochronous pendulum is

$$= \frac{b^2}{c} = \frac{(a-c)^2 + k^2}{c} = c - 2a + \frac{a^2 + k^2}{c}.$$

If the curve be not a circle, this result will still be true for small oscillations, if we take C the centre of curvature.

The point G may be at any distance from C, and hence it may be beyond the rolling surface PA; and thus the case treated in this problem includes the case of a body which hangs and oscillates, by means of an axis passing through it and resting at each end on a plane, as the pendulum in fig. 101.

PROB. VI. *When a pendulum is supported by an axis resting on a plane, to find the correction of the length of the pendulum due to the thickness of the axis.*

When it is requisite that a pendulum should oscillate very accurately about a horizontal line, it is constructed with an axis, as *B* in the pendulum *M*, fig. 101, of which the section is triangular; and this being placed with the ends upon different parts of a hard plane, and with the edge *O* downwards, the pendulum will turn round the axis *O*.

But, if the axis have a curvilinear section as *CP*, in the pendulum *N*, fig. 101; or, if the edge *O* be blunt, the pendulum no longer oscillates about a mathematical line, but oscillates supported by a rolling surface. In this case it is required to find from what *point* the pendulum must be suspended, that it may oscillate in the same time as when hanging by the blunt edge.

All parts of the pendulum being referred to the vertical plane in which the centre of gravity oscillates, let the body oscillate about a fixed point *S*, *C* being the centre of curvature; and let CS be $= \delta$, $CG = c$, and $CP = a$. Then, the length of the pendulum

$$= SG + \frac{k^2}{SG} = c - \delta + \frac{k^2}{c - \delta} = c - \delta + \frac{k^2}{c} + \frac{k^2 \delta}{c^2};$$

omitting powers of δ, because *CP*, and therefore *CS* is small.

Hence, that the time of this oscillation may be the same as the time of oscillation of the pendulum with the axis *CP*, we must have, by last Problem,

$$c - 2a + \frac{a^2 + k^2}{c} = c - \delta + \frac{k^2}{c} + \frac{k^2 \delta}{c^2}.$$

Hence, omitting $\frac{a^2}{c}$, which is small, we have

$$\delta = \frac{2ac^2}{c^2 - k^2}.$$

If $c > k$, the point S will lie towards G; if $c < k$, it will be beyond C. CS is always greater than CP.

Prob. VII. *A pendulum has two axes which are isochronous to each other, (as in* Art. 238.)*: supposing them cylindrical, to find the corresponding length of the simple pendulum.*

In Captain Kater's experiments, mentioned Art. 238, the pendulum was supposed to turn about a mathematical line; if instead of this, the pendulum oscillate on a blunt edge, the result will be found by the solution of this Problem.

Let c, c', be the distances of the centres of curvature of the two blunt axes from the centre of gravity of the pendulum; a, a' their radii. Then, since the pendulum oscillates in the same time about each axis, the length of the isochronous pendulum must be the same for both. Hence, by Prob. V.

$$l = \frac{(c'-a')^2+k^2}{c} = \frac{(c-a)^2+k^2}{c'}; \quad \therefore \; k^2 = \frac{c'(c-a)^2 - c(c'-a')^2}{c-c'};$$

$$\therefore \; l = \frac{(c-c')(c-a)^2 + (c-c')k^2}{(c-c')\,c} = \frac{(c-a)^2-(c'-a')^2}{(c-c')}$$

$$= c + c' - \frac{2(ca-c'a')}{c-c'} + \frac{a^2-a'^2}{c-c'}.$$

Cor. If we suppose a and a' to be equal, we have

$$l = c + c' - 2a,$$

the exact distance between the surfaces of the two axes.

Hence, the method of finding the length of the pendulum is equally correct, whether the edges be perfectly sharp or equally blunt.

If S, S', be the points from which the body would oscillate in the same time as it would do when hanging by the surfaces P, P', we have, in the case here supposed, $SS' = PP'$.

Sect. II. WHEN THE BODY MOVES IN ANY MANNER WHATEVER.

268. PROP. *The body being acted upon by any forces, the motion of the centre of gravity will be the same as if all those forces acted at that centre.*

If x', y', z', be the co-ordinates of a particle m; X, Y, Z, the forces on it; we shall have, as in last Section,

$$\text{effective forces, } \Sigma . m\frac{d^2x'}{dt^2},\quad \Sigma . m\frac{d^2y'}{dt^2},\quad \Sigma . m\frac{d^2z'}{dt^2};$$

$$\text{impressed forces, } \Sigma . mX,\quad \Sigma . mY,\quad \Sigma . mZ.$$

And as before, if $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the centre of gravity, $x' = \bar{x} + x$, &c., we have

$$\Sigma . m\frac{d^2x'}{dt^2} = \Sigma . m\frac{d^2\bar{x}}{dt^2} = M . \frac{d^2\bar{x}}{dt^2}, \text{ \&c.};$$

$$\therefore \frac{d^2x}{dt^2} = \frac{\Sigma . mX}{M};\quad \frac{d^2y}{dt^2} = \frac{\Sigma . mY}{M};\quad \frac{d^2z}{dt^2} = \frac{\Sigma . mZ}{M};$$

whence the proposition is true.

269. PROP. *The body being acted on by any forces, the motion of rotation will be affected in the same manner as if the centre of gravity were fixed, and the same forces were applied.*

This might be proved as in the corresponding proposition of last Section; but it appears also thus;—

Let the centre of gravity have a velocity V, and let the effect produced on its motion by the forces, be the same as if a single force P acted at the centre. Let new forces act so as to communicate to each point of the system a velocity equal and opposite to V. Then, the centre of gravity will be at rest; and the new forces which communicated these velocities

V will not affect the rotation about that centre, because their resultant will pass through that point[*]. Let there also be communicated to each point of the system a new force equal and opposite to P: then, the centre of gravity will have no tendency to move; and for the same reason as before, the rotation about that centre will not be affected by these other new forces.

Hence, if we suppose the centre of gravity to be fixed, and the same forces as before to act upon the body; the effect on the motion of rotation will not be altered.

Cor. 1. If the system be at rest when a force acts upon it, the instantaneous axis, about which it begins to revolve, must be perpendicular to the plane in which the line of force and the centre of gravity are. For otherwise the effective forces could not be equivalent to the force impressed, since the former would all be in planes parallel to each other, and oblique to the latter.

Cor. 2 In this case, the velocity communicated in the first instant is the same as if this instantaneous axis were fixed, and may be found by Art. 205.

Cor. 3. To find the circumstances of the motion of a body of any form, acted on by any forces, we must find the motion of the centre of gravity by the formulæ of Book I, and determine the motion of rotation by equations (θ), Art. 260.

270. Prob. I. *A homogeneous solid of revolution terminated by a sharp apex,* (PAB, fig. 102.) *moves so that the apex* (P) *is always upon a given horizontal plane; to determine its motion.*

Let PO be the axis of revolution; G the centre of gravity; let Gz_1, in the direction of PO, and Gx_1, Gy_1, perpendicular to PO, be moveable rectangular co-ordinates: and let Cx, Cy in the horizontal plane, and Cz perpendicular to it, be fixed rectangular co-ordinates.

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[*] If equal and parallel forces be communicated to each point of a system, their resultant will pass through the centre of gravity; for the force of gravity may be considered as a collection of equal and parallel forces acting on each point of the system, and the centre of gravity is the point through which the resultant passes in that case.

The forces which act on the body, are gravity (g), downwards at G, and the reaction of the plane (R), upwards at the point P. Let a'', and b'', as in Art. 254, be the cosines of the angles which the vertical line PK, or Cz, makes with Gx_1 and Gy_1; then, the components of R at P, parallel respectively to Gx_1 and Gy_1, will be Ra'' and Rb''; and if $GP = l$, the moments of the force R, with respect to the axes Gy_1 and Gx_1 will be $-Rla''$ and Rlb'' respectively. The moment with respect to Gz_1 will manifestly be 0.

If c'' be the cosine of the angle which PK or Cz makes with Pz_1, as in Art. 254, we have $GH = lc''$; and for the motion of the centre of gravity, M being the mass,

$$\frac{d^2 . lc''}{dt^2} = \frac{R}{M} - g;$$

$$\text{or } R = Mg + Ml\frac{d^2c''}{dt^2} \dots\dots\dots\dots\dots\dots (1).$$

And by equations (θ) for the motion of rotation, observing that $B = A$,

$$C\frac{dr}{dt} = 0 \dots\dots\dots\dots\dots\dots\dots (2),$$

$$A\frac{dq}{dt} + (A - C)\, rp = -Rla'' \dots\dots\dots\dots (3),$$

$$A\frac{dp}{dt} - (A - C)\, qr = Rlb'' \dots\dots\dots\dots\dots (4).$$

From (2) we have $r = e$, a constant quantity.

Also, taking (3) b'' + (4) a'', we have

$$A\left\{a''\frac{dp}{dt} + (b''r - c''q)p + b''\frac{dq}{dt} + (c''p - a''r)\, q\right\}$$

$$+ C\,(a''q - b''p)\, r = 0.$$

But we find as in Art. 262, p. 290, that

$$b''r - c''q = \frac{da''}{dt},\ c''p - a''r = \frac{db''}{dt},\ a''q - b''p = \frac{dc''}{dt};$$

substituting and integrating,

$$A(a''p + b''q) + Crc'' = k\dots\dots\dots(5).$$

Again, (3) q + (4) p gives

$$A\left(p\frac{dp}{dt} + q\frac{dq}{dt}\right) = Rl(pb'' - qa'').$$

Or, putting $-\frac{dc''}{dt}$ for $pb'' - qa''$, and for R its value from (1),

$$A\left(p\frac{dp}{dt} + q\frac{dq}{dt}\right) + Ml^2\frac{dc''}{dt}\frac{d^2c''}{dt^2} + Mlg\frac{dc''}{dt} = 0.$$

Multiply by 2, and integrate;

$$A(p^2 + q^2) + Ml\left(\frac{dc''}{dt}\right)^2 + 2Mlgc'' = h\dots\dots\dots(6).$$

Now let θ be the angle GPK; then $c'' = \cos.\theta, \frac{dc''}{dt} = -\sin.\theta\frac{d\theta}{dt}$.

And by Art. 261, we find

$$a'' = -\sin.\theta\sin.\phi,\ b'' = -\sin.\theta\cos.\phi;$$

and hence, by equations (ι) of that article

$$pa'' + qb'' = -\sin.^2\theta\frac{d\psi}{dt};\ p^2 + q^2 = \sin.^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2.$$

Hence, equations (5) and (6) become

$$\left.\begin{aligned} -A\sin.^2\theta\frac{d\psi}{dt} + Cr\cos.\theta = k \\ (A + Ml^2\sin.^2\theta)\frac{d\theta^2}{dt^2} + A\sin.^2\theta\frac{d\psi^2}{dt^2} + 2Mgl\cos.\theta = h \end{aligned}\right\}\dots\dots(7).$$

Eliminating $\frac{d\psi}{dt}$, we find

$$(A^2 \sin^2\theta + AMl^2 \sin.^4\theta)\frac{d\theta^2}{dt^2} = A\sin.^2\theta\,(h - 2Mgl\cos.\theta)$$
$$- (k - Cr\cos.\theta)^2 \ldots\ldots\ldots\ldots (8).$$

If we deduce from this the value of $\frac{dt}{d\theta}$, we shall find it $= F\theta$; $F\theta$ being a function of θ; and by integrating this, we have t in terms of θ, and θ in terms of t. And hence, by (7), we have ψ in terms of t; and hence, by equations (ι) we have ϕ. These integrations cannot be performed in finite terms.

The quantities h and k are to be determined from the given initial circumstances of the motion, by equations (7).

271. PROB. II. *The body having at first no motion except a rotation about its axis* PO; *to determine its motion afterwards.*

Let the original velocity about $PO = \epsilon$, and the original inclination $= \theta_1$. And since at first $\frac{d\psi}{dt} = 0$, $\frac{d\theta}{dt} = 0$, we have

$r = \frac{d\phi}{dt} = \epsilon$; and by equations (7), of last article,

we have at first $C\epsilon\cos.\theta_1 = k$, $2Mgl\cos.\theta_1 = h$.

Hence, equation (8) becomes

$$(A^2 \sin.^2\theta + AMl^2 \sin.^4\theta)\frac{d\theta^2}{dt^2}$$
$$= 2AMgl\sin.^2\theta\,(\cos.\theta_1 - \cos.\theta) - C^2\epsilon^2(\cos.\theta_1 - \cos.\theta)^2.$$

The expression on the right-hand side consists of two factors; $\cos.\theta_1 - \cos.\theta$, and (putting $1 - \cos.^2\theta$ for $\sin.^2\theta$),

$$2AMgl - C^2\epsilon^2\cos.\theta_1 + C^2\epsilon^2\cos.\theta - 2AMgl\cos.^2\theta.$$

If we put $\frac{C^2}{4AMgl} = m$, and

$$\alpha = m\epsilon^2 - \sqrt{(1 - 2m\epsilon^2 \cos.\theta_1 + m^2\epsilon^4)},$$
$$\beta = m\epsilon^2 + \sqrt{(1 - 2m\epsilon^2 \cos.\theta_1 + m^2\epsilon^4)};$$

the above expression becomes

$$2AMgl(\cos.\theta - \alpha)(\beta - \cos.\theta).$$

Hence, we shall have

$$\frac{d\theta^2}{dt^2} = \frac{2Mgl(\cos.\theta_1 - \cos.\theta)(\cos.\theta - \alpha)(\beta - \cos.\theta)}{\sin.^2\theta(A + Ml^2\sin.^2\theta)}.$$

It is easily seen that α is less than $\cos.\theta_1$, and that β is greater than 1; let $\alpha = \cos.\theta_2$; and θ_2 and θ_1, will be the greatest and least values of θ. The axis will oscillate perpetually between these inclinations, and the rotation will continue for ever.

If *PO* be inclined beyond a certain limit, the sides of the body *POB* will touch the horizontal plane, and physically speaking, the rotation can no longer continue in the same manner. If we suppose that *PO* is susceptible of all positions above a horizontal one, without the rotation being thus stopped, it will go on so long as θ_2 is less than a right angle: that is, if $\cos.\theta_2$ or

$$\alpha > 0, \text{ or if } m\epsilon^2 - \sqrt{(1 - 2m\epsilon^2\cos.\theta_1 + m^2\epsilon^4)} > 0,$$

$$\text{if } 2m\epsilon^2\cos.\theta_1 > 1; \text{ if } \epsilon^2 > \frac{1}{2m\cos.\theta_1}, \text{ or } \epsilon^2 > \frac{2AMgl}{C^2\cos.\theta_1}.$$

If ϵ, the original velocity, be less than this, the body will fall to a horizontal position.

If $\theta_1 = \theta_2$, the axis will always retain the same inclination. This supposes

$$\cos.\theta_1 = m\epsilon^2 - \sqrt{(1 - 2m\epsilon^2\cos.\theta_1 + m^2\epsilon^4)},$$

and cannot be except either ϵ, the velocity of rotation, be infinite, or $\theta_1 = 0$: in the latter case the axis is always vertical.

272. **Prob. III.** *The body and the axis having velocities communicated to them; to determine the conditions in order that the axis* PO *may always retain the same inclination.*

In this case θ, and therefore c'', is constant; and by equation (1) of Art. 270, $R = Mg$. Also, by (7), $\frac{d\psi}{dt}$ will be constant; let it $= \delta$.

$$\text{Then, } p = \sin.\theta \sin.\phi \frac{d\psi}{dt} = -a''\delta;$$

$$\therefore a'' = -\frac{p}{\delta}; \text{ similarly, } b'' = -\frac{q}{\delta};$$

and equations (3), (4) become

$$A\frac{dq}{dt} + \left(Ar - Cr + \frac{Mgl}{\delta}\right) p = 0,$$

$$A\frac{dp}{dt} - \left(Ar - Cr + \frac{Mgl}{\delta}\right) q = 0.$$

Hence, we find as in p. 295, making $\frac{(A-C)r\delta + Mgl}{A\delta} = \epsilon$,

$$p = a\sin.(\epsilon t + \gamma), \quad q = a\cos.(\epsilon t + \gamma).$$

Substituting in the first and second of equations (ι), and dividing, we have

$$\tan.(\epsilon t + \gamma) = \tan.\phi; \quad \epsilon t + \gamma = \phi, \quad \frac{d\phi}{dt} = \epsilon.$$

And by the third of equations (ι), $r = \epsilon - \cos.\theta\,.\,\delta$.

$$\text{But } r = \frac{A\delta\epsilon - Mgl}{(A-C)\delta}.$$

$$\text{Hence, } C\delta\epsilon + (A-C)\cos.\theta\,.\,\delta^2 = Mgl.$$

Hence, δ and θ being known, ϵ is known.

If $\delta = 0$, we have ϵ infinite, agreeing with the last Problem.

The results in the preceding Problems are applicable to the motion of a *spinning top*, considering it as upon a perfectly smooth plane, and supported on a mathematical apex. It has appeared, that under these circumstances the top will, if a sufficient velocity be communicated to it, go on revolving for ever; but, in consequence of the absence of friction, the motion will be a good deal different from that which we observe actually to take place in such cases. The centre of gravity in our Problem either remains at rest, or moves up and down in a vertical line, and cannot have any curvilinear motion. The axis can never become more nearly vertical than it was at the beginning of the motion, though it will at intervals return to the inclination which it then had. But in the experiment, the top, if inclined at first, will approach to a vertical position, which it will, as near as the senses can judge, attain and preserve for some time; and the centre of gravity will frequently describe a curve approaching to a circle, while the apex or foot of the instrument remains stationary. These differences of theory and practice appear to be attributable to the effects of friction*.

273. A Problem of great consequence, depending on the principles of this Chapter, is that of the *Precession of the Equinoxes*, or motion of the nodes of the Earth's equator on the ecliptic, and similar motions in the heavenly bodies. This motion in the Earth arises from the attraction of the Sun on the Earth, which, in consequence of the spheroidal form of the

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* Euler thus explains the effect of friction in causing a top to raise itself into a vertical position. "The friction will perpetually retard the motion of the apex P of the instrument, and at last reduce it to rest. If this happen before the top fall, it must then be spinning in such a position, that the point can remain stationary. But this cannot be if it be inclined. Hence, it must have a tendency to erect itself into a vertical position." *Theor. Mot. Corp. Solid et Rigid*, Suppl. Cap. 4.

This property has been applied to obtain an artificial horizon. Since the axis tends to become vertical, a plane perpendicular to the axis tends to become horizontal; and for a considerable length of time may be considered as accurately so. See *Phil. Trans.* 1752.

Mr Landen's solution of this Problem appears to contain several mistakes. See *Mathematical Lucubrations.*

latter, produces a force tending to turn its poles, one towards, and one from, the Sun. And this force, combined with the Earth's rotation, produces a motion like that described in p. 296. If, in fig. 96, xy be the fixed plane of the ecliptic, x_1y_1 the Earth's equator, and z_1 its north pole, N, the intersection of xy and x_1y_1, moves along xz in a direction opposite to the diurnal rotation. The investigation of this subject belongs to Physical Astronomy; but, in order to shew the nature of the action exercised, we shall take the inverse Problem, the motion being given to find the forces.

PROB. IV. *A figure of revolution, turning uniformly on its axis, retains the same inclination, while the nodes of its equator move uniformly on a fixed plane; to find the forces by which it is acted on.*

The notation remaining as in Art. 261, fig. 95, let

$$\frac{d\phi}{dt} = \epsilon, \text{ the velocity of rotation;}$$

$$\frac{d\psi}{dt} = \delta, \text{ the velocity of the nodes; } \theta \text{ constant.}$$

Hence, by equations (ι),

$$p = \delta \sin.\phi \sin.\theta, \quad q = \delta \cos.\phi \sin.\theta, \quad r = \epsilon - \delta \cos.\theta,$$
$$\text{and } B = A.$$

Hence, since δ, ϵ are constant, we have, by equations (θ),

$$\begin{aligned} L &= A\frac{dp}{dt} + (C - A)qr \\ &= A\delta\epsilon \cos.\phi \sin.\theta + (C - A)(\epsilon - \delta\cos.\theta)\delta\cos.\phi \sin.\theta \\ &= \{C\epsilon + (A - C)\delta\cos.\theta\}\delta\cos.\phi\sin.\theta \end{aligned}$$

$$\begin{aligned} M &= A\frac{dq}{dt} + (A - C)pr \\ &= -A\delta\epsilon \sin.\phi \sin.\theta + (A - C)(\epsilon - \delta\cos.\theta)\delta\sin.\phi\sin.\theta \\ &= -\{C\epsilon + (A - C)\delta\cos.\theta\}\delta\sin.\phi\sin.\theta \end{aligned}$$

$$N = C\frac{dr}{dt} = 0.$$

Hence, if $F = \{C\epsilon + (A - C)\delta \cos.\theta\}\delta \sin.\theta$,

$$L = F\cos.\phi,\quad M = -F\sin.\phi,\quad N = 0.$$

These forces L, M, N will be supplied, if we suppose a force F to act any where in the circle zz_1, fig. 95, urging z_1 towards z, or urging the plane Nx_1y_1 towards the plane Nxy. For by p. 287, L, M, N are the moments of F about the axes x_1, y_1, z_1, respectively; and since $\phi =$ angle Nz_1x_1, a force F acting at z_1 in the direction $z_1\, z_1$ will be equivalent to two forces, $F\sin\phi$ in the direction $y_1\, z_1$ and $F\cos\phi$ in the direction $z_1\, x_1$.

Cor. Therefore a constant force perpetually urging the plane x_1y_1 towards a fixed plane, will produce such a motion as is here supposed; that is, it will make the body move so that its axis of rotation retains the same inclination to a fixed plane, while the nodes of its equator move with a uniform motion in this plane.

The motion of the Earth which produces the precession of the equinoxes is of this kind, if we abstract the nutation. Hence, such a precession would be produced by a force which should perpetually urge the equator of the revolving body towards the fixed plane.

APPENDIX.

THE MOTION OF FLEXIBLE AND OF ELASTIC BODIES.

274. THE problem of the motions of flexible and elastic bodies is of too complex a nature to be treated in its general form; but particular cases have been the subject of mathematical investigation. Of these we shall take two examples.

OF THE VIBRATIONS OF STRINGS.

PROB. I. *A uniform string being stretched between two given points; to find the time of its small vibrations.*

Let a be the length of the string between the fixed points; Wg its weight, and Fg the force by which it is stretched. It is supposed that the vibrations are exceedingly small, so that the length, and consequently the tension, may remain the same in all the forms which the curve assumes. Let APB, fig. 103, be the form at a time t, and let AM, MP, an abscissa and ordinate, be x and y, and $AP = s$. Let AN, NQ be another abscissa and ordinate, and $PQ = h$. The tension will be the same throughout, and $= Fg$; hence, at P the part of it resolved in the direction PM is $Fg \cdot \frac{dy}{ds}$.

$$\text{At } Q, \ \frac{dy}{ds} \text{ becomes } \frac{dy}{ds} + \frac{d^2y}{ds^2} \cdot h + \frac{d^3y}{ds^3} \cdot \frac{h^2}{1 \cdot 2} + \&\text{c.},$$

Hence, the difference of the forces at P, Q, that is, the moving force by which PQ is drawn in the direction MP, is

$$Fg \cdot \left\{ \frac{d^2y}{ds^2} h + \frac{d^3y}{ds^3} \cdot \frac{h^2}{1 \cdot 2} + \&\text{c.} \right\}.$$

And since the weight of a length of the string a is Wg, the weight of a length h is $\frac{Wgh}{a}$; hence, the mass of PQ is $\frac{Wh}{a}$, and the accelerating force on PQ is

$$\frac{Fag}{W}\left\{\frac{d^2y}{ds^2}+\frac{d^3y}{ds^3}\cdot\frac{h}{2}+\&\text{c.}\right\}.$$

And when PQ is taken very small, every part of it moves with the same velocity, and this is the true expression for the accelerating force; in this case h vanishes.

That is, accelerating force on P, in direction $MP=\frac{Fag}{W}\cdot\frac{d^2y}{ds^2}$.

Since the vibrations are very small, we may suppose P to move always perpendicularly to AM, that is, in MP; and hence, the accelerating force on $P=\frac{d^2y}{dt^2}$. Also, for the same reason, we may suppose s to be equal to x. Hence, we have

$$\left(\frac{d^2y}{dt^2}\right)=\frac{Fag}{W}\left(\frac{d^2y}{dx^2}\right)\ldots\ldots(1).$$

The quantities are put in brackets to indicate that they are *partial* differential coefficients; in the one, y is differentiated, supposing t only to vary, and in the other, y is differentiated supposing x only to vary. The former differentiation refers to a change in the position of a given point of the curve; the latter, to a passage from one point of the curve to another at a given moment of time.

The ordinate y will be a function of x and t, which is to be determined by integrating the partial differential equation (1). Its integration is given, Lacroix, *Elem. Treat.* Art. 319; and if $\frac{Fag}{W}=b^2$, it is

$$y=\phi(x+bt)+\psi(x-bt)\ldots\ldots\ldots(2).$$

It is evident from trial, that this satisfies equation (1), for if we differentiate twice with respect to x, we find

$$\left(\frac{d^2y}{dx^2}\right)=\phi''(x+bt)+\psi''(x-bt),$$

where $\phi''z$ indicates $\frac{d^2\phi z}{dz^2}$. Also, differentiating with respect to t, we have

$$\left(\frac{d^2y}{dt^2}\right) = b^2\phi''(x+bt) + b^2\psi''(x-bt).$$

And it is manifest, that these values verify equation (1). The form of the functions ϕ and ψ is to be determined from the initial circumstances of the string.

Differentiating equation (2), we have

$$\frac{dy}{dt} = b\left\{\phi'(z+bt) - \psi'(x-bt)\right\}.$$

Also if we suppose that the curve was at rest in any position when $t=0$, we must have $\phi'(x) - \psi'(x) = 0$. Hence, $\psi'(x) = \phi'(x)$, and $\psi(x) = \phi(x)$. Let $\phi(x) = \frac{1}{2}f(x)$, and equation (2) becomes

$$y = \tfrac{1}{2}\left\{f(x+bt) + f(x-bt)\right\} \ldots\ldots (3),$$

$$\therefore\ \frac{dy}{dt} = \frac{b}{2}\left\{f'(x+bt) - f'(x-bt)\right\} \ldots\ldots (4).$$

Equation (3) gives the position of the curve at any time, and equation (4) the velocity of any point; b is equal to $\sqrt{\frac{Fag}{W}}$.

If we make $t=0$, we have $y=f(x)$, which is the equation to the original form of the string, and hence, the form of f is known. The original form may be any whatever, and f any function whatever, subject to the conditions that it must $=0$, when $x=0$, and when $x=a$. It appears from the theory of partial differential equations, that the function f, introduced in integration, may be *discontinuous;* that is, the initial form of the string *APB*, may be composed of different lines, not expressed by the same equation all the way from *A* to *B*.

275. From the nature of the reasoning, we obtain other properties of the function f. The points *A*, *B*, are fixed; hence, we must have $y=0$, when $x=0$, whatever be t. Therefore by equation (3),

$$0 = f(bt) + f(-bt);$$

or, if we make $bt=u$, $f(-u) = -f(u) \ldots\ldots\ldots\ldots\ldots (5)$.

Also, if $x = a$, $y = 0$, whatever be t;

$$\therefore\ 0 = f(a+u) + f(a-u), \text{ and } f(a+u) = -f(a-u) \ldots (6).$$

From equation (5) it appears, that the curve represented by $y = f(x)$, is continued with similar forms on each side of A, the curve being on one side above, and on the other below the axis. Also, from equation (6) it appears, that the curve is continued with similar forms on each side of B, above the axis on one side, and below it on the other. Hence, the curve is continued indefinitely on both directions, in the manner represented in fig. 104, and it is the ordinate of this curve which is to be considered as $f(x)$ in finding y from equation (3).

To find the position of a point P at the end of any time t, we must take $MN = MN' = bt$, and $MQ = \frac{1}{2}(NO + N'O')$ will give Q, the position of P after a time t. And as the curve is continued indefinitely, the time may be supposed to increase indefinitely, and the same construction will always determine the position.

The string will perform oscillations; that is, it will assume a position similar to its original one, on the opposite side of the string, return to its original position, and so on perpetually; as may thus be shewn:

By (6), $f(a+u) = -f(a-u)$. Let $u - a = u'$; $\therefore\ u = a + u'$,

$$f(2a+u') = -f(-u') = f(u') \text{ by (5), or } f(2a+u) = f(u):$$

similarly, $f(4a+u) = f(u)$, and generally,

$$f(2ma+u) = f(u), \ m \text{ being any whole number.}$$

Hence, the ordinate for an abscissa $2ma + x$ is the same as for x, as also appears from fig. 104.

If, therefore, we assume $bt = 2a$, $bt = 4a, \ldots bt = 2ma$, at the times corresponding to each of these values, we shall have $y = \frac{1}{2}\{f(x) + f(x)\} = f(x)$, and every point of the curve is in its original position. The intervals between these times are the lengths of complete vibrations of the string, and we have, for these intervals,

$$bt = 2a, \quad t = \frac{2a}{b} = 2\sqrt{\frac{Wa}{Fg}}.$$

If the thickness and material of the string be given, W varies as a. Hence, *for a given string, the time of vibration is as the length directly, and as the square root of the tension inversely.*

If we make $bt = a$, or $t = \frac{b}{a}$, we have

$$y = \tfrac{1}{2}\{f(x+a) + f(x-a)\}.$$

And $f(x-a) = -f(a-x)$ by (5), $= f(a+x)$ by (6);

$$\therefore\ y = f(x+a).$$

And hence, after the time $t = \frac{b}{a}$, the form of the curve is ARB, in fig. 103, identical with BDB' in fig. 104, and exactly similar to the original curve inverted; the greatest ordinate being now at the same distance from B, as it was from A at the beginning of the motion. Similarly, at the end of times $\frac{3b}{a}$, $\frac{5b}{a}$, &c., the figure will be the same as ARB.

If we make $bt = \frac{a}{2}$, we have the figure in the middle of the time between the positions APB and ARB.

Hence, at this time, $y = \frac{1}{2}\left\{f\left(x + \frac{a}{2}\right) + f\left(x - \frac{a}{2}\right)\right\}$.

But $f\left(x - \frac{a}{2}\right) = -f\left(\frac{a}{2} - x\right)$ by (5);

$$\therefore\ y = \tfrac{1}{2}\left\{f\left(\frac{a}{2} + x\right) - f\left(\frac{a}{2} - x\right)\right\},$$

$\frac{a}{2} + x$, and $\frac{a}{2} - x$ correspond to points at equal distances from

C, the middle point of *AB*. If in the original form of the curve, the ordinates for the portion *CB* be greater than for *AB*, the position of the portion *AEC* at the middle of the time of oscillation will be above the axis *AC*. In the same manner it may be shewn, that the position of the portion *CFB* at that time will be below the axis *CB*, and similar to the portion *AEC* inverted. The string never becomes a straight line.

If the curve, instead of being at first in the position *APB*, have an original position *APBDB'*, fig. 104, *A* and *B'* being the extreme points, and the curve consisting of two equal and similar portions *APB*, *B'DB*, it will oscillate so that the point *B* will remain fixed; each half, *APB*, *BDB'* will oscillate as if the string were fixed at *A*, *B*, and at *B*, *B'*, and the vibrations will employ half the time. Similarly, if the original form be *APBDB'B''*, the points *B* and *B'* will remain fixed, and the oscillations will employ one-third of the time, and so on. The points *B*, *B'* which remain fixed, are sometimes called *nodes*.

276. The musical tone, or *note*, produced by a vibrating string, depends upon the rapidity of the vibrations. If a string *a* vibrate twice as fast as another *A*, the note produced by *a* is said to be an *octave above* that produced by *A*. If a string *e* vibrate three times, while *a* vibrates twice, the note of *e* is a *fifth above* that of *a*; and so on. And notes whose numbers of vibrations are to each other in ratios so simple as these, are found, when combined, to be agreeable to the ear.

It appears from what has been said, that a string may vibrate so that no point of it is at rest, except its two extremities, and thus give the *fundamental note*; or it may vibrate so that its middle point is at rest, in which case it will produce the octave to the fundamental note; or it may vibrate so as to have two points at rest, dividing it into three equal parts, and it will then give the fifth above the octave; and so on.

It appears by experiment, that a wire which performs 240 vibrations in a second, sounds the note called *C* in the middle of the scale of a harpsichord. It has been proposed however to consider 256, or 2^8, as the number of vibrations determining this note, in which case every other note which is called *C* would also have a power of 2 for the number of its vibrations. We may find the note produced by any string, by finding the number of its vibrations, and comparing them with the number just mentioned.

277. Prob. II. *To find the form of a string, that it may oscillate symmetrically, that is, that all its points may come to the axis at the same time.*

In this case, the force which accelerates each particle, must be as its distance from the axis; that is, as *y*. Hence, taking the expression for the force obtained in p. 328, we have

$$\frac{Fag}{W} \cdot \frac{d^2y}{dx^2} = -my, \text{ } m \text{ being a constant quantity,}$$

$$\text{or } \frac{d^2y}{dx^2} + \frac{Wm}{Fag} \cdot y = 0.$$

Of this, the integral is, if $\frac{Wm}{Fag} = k^2$,

$$y = C \sin. kx + C' \cos. kx;$$

$$\therefore \frac{dy}{dx} = kC \cos. kx - kC' \sin. kx.$$

$$\text{And when } x = 0, \ \frac{dy}{dx} = 0; \ \therefore C' = 0,$$

$$y = C \sin. kx.$$

The quantity *C* is arbitrary, and determines the magnitude of the original ordinate. It must necessarily be small, because the vibrations are supposed to be small.

The time of a point coming to the axis will be $\frac{\pi}{2\sqrt{m}}$, and hence, the time of an oscillation to the opposite side, is

$\frac{\pi}{\sqrt{m}}$, and the time of a complete vibration, the string returning into the original position, is $\frac{2\pi}{\sqrt{m}}$.

The curve must meet the axis again when $x = a$, hence,

$$0 = C \,.\, \sin. ka\,; \;\therefore\; ka = \pi, \text{ and } k = \frac{\pi}{a}\,;$$

$$\therefore\; \sqrt{\frac{Wm}{Fag}} = \frac{\pi}{a}, \text{ and } \sqrt{m} = \pi\sqrt{\frac{Fg}{Wa}}.$$

Hence, the time of a vibration $= \frac{2\pi}{\sqrt{m}} = 2\sqrt{\frac{Wa}{Fg}}$, the same as before.

ON THE VIBRATIONS OF SPRINGS.

278. As another case of small oscillations, we shall obtain the equations for the motion of an elastic rod.

PROP. I. *A uniform elastic rod* BC, fig. 105, *firmly fixed at* B, *and naturally straight, vibrates by its elasticity; to find the equations of its motion in small vibrations.*

We shall find first the forces which must act at every point to keep the rod *at rest* in the position BPC. Let AB be its position when left to itself, AN, NQ, an abscissa and ordinate; $CQ = s$. And suppose that a force F, acting at every point Q, keeps the rod $BPQC$ in equilibrium. Since the whole is in equilibrium, the part PQC must be so; and hence, the forces which act on PC must balance the tendency which the rod has to straight itself at P by its elasticity. Now the moment of the elasticity is directly as the curvature, and is therefore inversely as the radius of the curvature at P. Let this radius be ρ; and let the effect of the elasticity to turn PC round P be that of a force K acting at any arm c. Hence,

$$Kc \propto \frac{1}{\rho}. \quad \text{Let } Kc = \frac{E}{\rho}.$$

And, if $AM = x'$, $MP = y'$, $AN = x$, $NQ = y$, the moment of any force F about P is $F(x' - x)$; and, as the forces F act at every point of PC, the moment of the forces on a small element ds is $F(x' - x)\,ds$, or $F(x' - x)\,dx$, when the flexure is very small, so that dx and ds may be conceived to coincide. Hence, $\int_x F(x' - x)$ is the whole moment of the forces on PC, the integral being taken from $x = 0$ to $x = x'$. Therefore,

$$\frac{E}{\rho} = \int_x . F(x' - x).$$

$$\text{Now, } \frac{1}{\rho} = -\frac{\frac{d^2y'}{dx'^2}}{\frac{ds'^3}{dx'^3}} = -\frac{d^2y'}{dx'^2}, \text{ nearly.}$$

And $\int_x F(x' - x) = x' \int_x F - \int_x Fx$.

If we suppose $\int_x F$ taken from $x = 0$, to be $= F_1$, we shall have

$$\int_x Fx = F_1 x - \int_x F_1.$$

Hence, $\int_x F(x' - x)$ becomes $\int_x F_1$. And we have

$$-E\frac{d^2y'}{dx'^2} = \int_x F_1;$$

$$\therefore \ -E\frac{d^3y'}{dx'^3} = F_1;$$

for it is clear that $\int_x F_1$, from $x = 0$ to $x = x'$, will be a function of x', and that its differential coefficient with respect to x' will be F_1. Also

$$-E\frac{d^4y'}{dx'^4} = \frac{dF_1}{dx} = F,$$

on the same account. Hence, F is known at every point.

Now if, when the spring is kept at rest by these forces F acting at every point, we suppose at each point a force equal to F to act in the opposite direction, these opposite forces will counterbalance each other, and the rod will be left to its own elasticity. But, in this case the points are manifestly each urged *from rest* by a force F. Hence, we shall have $\frac{d^2y}{dt^2} = F$; or, using the same notation as before,

$$-E\left(\frac{d^4y}{dx^4}\right) = \left(\frac{d^2y}{dt^2}\right)\text{............}(1),$$

which is the equation to the motion. This cannot be integrated in finite terms, and though we might obtain integrals in series, we shall confine ourselves to the consideration of symmetrical oscillations.

279. Prop. II. *To find the form of an elastic rod, that it may vibrate symmetrically.*

For this purpose, the force which urges each point towards the axis must be as the distance from the axis. Hence, we must have

$$E\frac{d^4y}{dx^4} = my, \text{ or } \frac{d^4y}{dx^4} = \frac{my}{E} = k^4y \text{ suppose.........}(2).$$

This equation may be integrated, and gives

$$y = A\cos. kx + B\sin. kx + C\epsilon^{kx} + D\epsilon^{-kx}\text{.........}(3).$$

A, B, C, D, being arbitrary quantities. To determine them, we may observe, that when $x = 0$, we must have $\frac{d^3y}{dx^3} = 0$, because $\frac{d^3y}{dx^3} = F_1$ in last Article. For the same reason, $\frac{d^2y}{dx^2} = 0$, when $x = 0$. Hence,

$$0 = -B + C - D,\ 0 = -A + C + D;$$

$$\therefore\ A = C + D, \qquad B = C - D.$$

Again, since the extremity B is fixed, we have, when $x = a$, the whole length, $y = 0$. And, since that extremity is also fixed in direction, we have $\frac{dy}{dx} = 0$, when $x = a$. Hence, we find

$$0 = \quad (C+D)\cos. ka + (C-D)\sin. ka + C\epsilon^{ka} + D\epsilon^{-ka},$$

$$0 = -(C+D)\sin. ka + (C-D)\cos. ka + C\epsilon^{ka} - D\epsilon^{-ka};$$

whence

$$\frac{C}{D} = \frac{\sin. ka - \cos. ka - \epsilon^{-ka}}{\sin. ka + \cos. ka + \epsilon^{ka}} = \frac{\cos. ka + \sin. ka + \epsilon^{-ka}}{\cos. ka - \sin. ka + \epsilon^{ka}}.$$

Hence, we find $2 + \cos. ka\,(\epsilon^{ka} + \epsilon^{-ka}) = 0 \ldots\ldots\ldots (4).$

From this equation k is to be determined: k will have an infinite number of different values; and to each of these will correspond a different form of the curve, determined by equation (3), in which form the rod will oscillate symmetrically.

The least positive value of ka is 1,8751, nearly. The other values of ka are very nearly $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, &c.

280. The time of a vibration will be $\frac{2\pi}{\sqrt{m}}$. Now $\frac{m}{E} = k^4$, whence $m = Ek^4$. And hence, the time of vibration is

$$\frac{2\pi}{k^2\sqrt{E}} = \frac{2\pi a^2}{k^2 a^2 \sqrt{E}}.$$

For the same value of ka, that is, for similar forms, in rods of different lengths, it appears that *the time of the vibration is as the square of the length directly, and as the square root of the elasticity inversely.*

Hence, if a rod A' were twice as long as A, and of the same elasticity, the note produced by A' would be four octaves below that produced by A.

The different values of ka in equation (4), taken for the same rod, give a series of different forms, resembling those in fig. 22, and differing like them in the number of their nodes. The times of vibration will be successively as the values

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of $\frac{1}{k^2 a^2}$; and the numbers of vibrations in a given time, and consequently, the note, will be as $k^2 a^2$. Hence, the note produced by the successive figures will be as

$$1; \quad 6,2673; \quad 17,549; \quad 34,386; \text{ \&c.}$$

Hence, the notes which an elastic rod can produce, besides the fundamental note, are, the fifth above the double octave, somewhat too sharp: the half note above the fourth octave nearly: the half note above the fifth octave nearly: and so on.

If the rod be not fixed into a support at B, but be subject to other conditions, for instance, if both ends were fastened, or both free, we shall have to make some other supposition instead of putting $y = 0$, and $\frac{dy}{dx} = 0$, when $x = a$. The rest of the process will be nearly the same as before.

We may find E by the equation $E = Kc\rho$, where a force K, acting at an arm c, produces a radius of curvature ρ. See *Statics*.

For an examination of the different cases of this Problem, see *Com. Acad. Petrop.* for 1741—1743, p. 105, and *Novi Com. Acad. Petrop.* for 1772, p. 449. Dr Young, *Elem. of Nat. Phil.* Art. 398. For experiments, see Chladni, *Traité d'Acoustique*, p. 92; Biot, *Traité de Physique*, Tom. II, p. 74.

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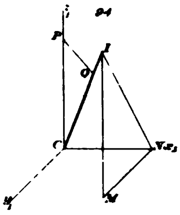

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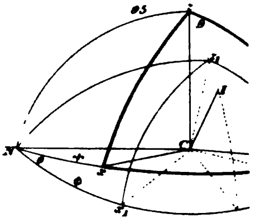

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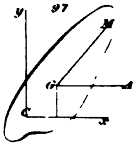

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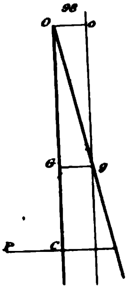

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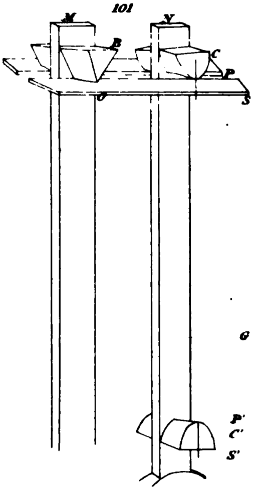

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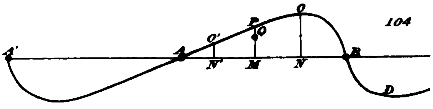

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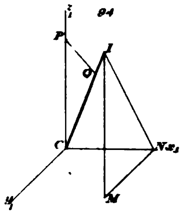
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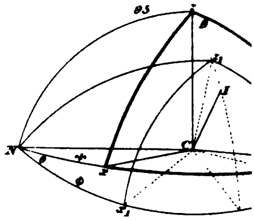
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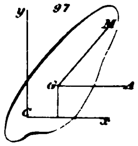
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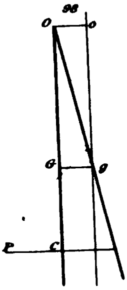
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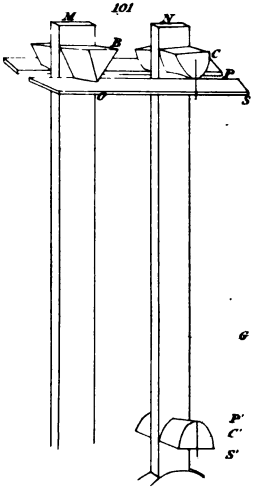
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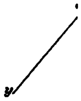
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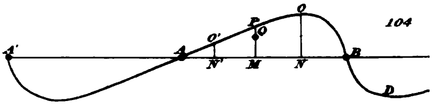
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